PONDEROMOTIVE INSTABILITIES
AND MICROPHONICS

A TUTORIAL

Jean Delayen

Thomas Jefferson National Accelerator Facility
Some Definitions

- Ponderomotive effects: changes in frequency caused by the electromagnetic field (radiation pressure)
  - Static Lorentz detuning (cw operation)
  - Dynamic Lorentz detuning (pulsed operation)

- Microphonics: changes in frequency caused by connections to the external world
  - Vibrations
  - Pressure fluctuations

Note: The two are not completely independent.
When phase and amplitude feedbacks are active, ponderomotive effects can change the response to external disturbances
Some History

- M. M. Karliner, V. E. Shapiro, I. A. Shekhtman
  *Instability in the Walls of a Cavity due to Ponderomotive Forces of the Electromagnetic Field*
  Soviet Physics – Technical Physics, Vol. 11, No. 11, May 1967

- V. E. Shapiro
  *Ponderomotive Effects of Electromagnetic Radiation*

- M. M. Karliner, V. M. Petrov, I. A. Shekhtman
  *Vibration of the Walls of a Cavity Resonator under Ponderomotive Forces in the Presence of Feedback*

Stability conditions derived by comparing the rate of transfer of energy from the electromagnetic mode to the mechanical mode with the dissipation rate of the mechanical mode.
Analysis valid when decay time of the electromagnetic mode much less than period of mechanical mode (\(\tau \Omega_\mu << 1\)), or when the rate of transfer of energy is very high.
Some More History

- D. Schulze
  *Mechanical Instabilities of a Superconducting Helical Structure due to Radiation Pressure*
  Proc. 1970 Proton Linac Conference, Batavia

_Ponderomotorische Stabilität von Hochfrequenzresonatoren und Resonatorregelungssystemen_
Dissertation, KFK-1493, December 1971

Analysis of stability of generator-driven superconducting resonator (arbitrary \( \tau \Omega_{\mu} \)) using control systems methods (Laplace transforms, transfer functions) with and without phase and amplitude feedback
Analysis of damping of mechanical vibrations by ponderomotive forces
Some More History

- J. R. Delayen
  *Phase and Amplitude Stabilization of Superconducting Resonators*
  Dissertation, Caltech, 1978

Analysis of ponderomotive instabilities of resonators operated in self-excited loops, with and without feedback, using control systems methods.

Included I-Q detection

Used methods of stochastic analysis to quantify performance of feedback system in the presence of microphonics and ponderomotive effects

Later extended to include effects due to presence of beam loading and electronic damping.
Ponderomotive Effects

- Adiabatic theorem applied to harmonic oscillators (Boltzmann-Ehrenfest)

If \( \varepsilon = \frac{1}{\omega^2} \frac{d\omega}{dt} \ll 1 \), then \( \frac{U}{\omega} \) is an adiabatic invariant to all orders

\[
\Delta \left( \frac{U}{\omega} \right) / \left( \frac{U}{\omega} \right) \sim o(e^{-d/\varepsilon}) \quad \Rightarrow \quad \frac{\Delta \omega}{\omega} = \frac{\Delta U}{U} \quad \text{(Slater)}
\]

Quantum mechanical picture: the number of photons is constant: \( U = N\hbar\omega \)

\[
U = \int_V \, dv \left[ \frac{\mu_0}{4} H^2(\vec{r}) + \frac{\varepsilon_0}{4} E^2(\vec{r}) \right] \quad \text{(energy content)}
\]

\[
\Delta U = -\int_S \, dS \, \vec{n}(\vec{r}) \cdot \vec{\xi}(\vec{r}) \left[ \frac{\mu_0}{4} H^2(\vec{r}) - \frac{\varepsilon_0}{4} E^2(\vec{r}) \right] \quad \text{(work done by radiation pressure)}
\]
Ponderomotive Effects

\[
\frac{\Delta \omega}{\omega} = - \frac{1}{s} \int dS \bar{n}(\vec{r}) \cdot \vec{\xi}(\vec{r}) \left[ \frac{\mu_0}{4} H^2(\vec{r}) - \frac{\varepsilon_0}{4} E^2(\vec{r}) \right]
\]

\[
\int_v d\nu \left[ \frac{\mu_0}{4} H^2(\vec{r}) + \frac{\varepsilon_0}{4} E^2(\vec{r}) \right]
\]

Expand wall displacements and forces in normal modes of vibration \( \phi_\mu(\vec{r}) \) of the resonator

\[
\int_S dS \phi_\mu(\vec{r}) \phi_\nu(\vec{r}) = \delta_{\mu\nu}
\]

\[
\xi(\vec{r}) = \sum_\mu q_\mu \phi_\mu(\vec{r}) \quad q_\mu = \int_S \xi(\vec{r}) \phi_\mu(\vec{r}) \, dS
\]

\[
F(\vec{r}) = \sum_\mu F_\mu \phi_\mu(\vec{r}) \quad F_\mu = \int_S F(\vec{r}) \phi_\mu(\vec{r}) \, dS
\]


Ponderomotive Effects

Equation of motion of mechanical mode $\mu$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\mu} - \frac{\partial L}{\partial q_\mu} + \frac{\partial \Phi}{\partial \dot{q}_\mu} = F_\mu$$

$L = T - U$ (Euler-Lagrange)

$$U = \frac{1}{2} \sum_\mu c_\mu q_\mu^2$$ (elastic potential energy) $\quad c_\mu$: elastic constant

$$T = \frac{1}{2} \sum_\mu c_\mu \frac{\dot{q}_\mu^2}{\Omega_\mu^2}$$ (kinetic energy) $\quad \Omega_\mu$: frequency

$$\Phi = \sum_\mu \frac{c_\mu \dot{q}_\mu^2}{\tau_\mu \Omega_\mu^2}$$ (power loss) $\quad \tau_\mu$: decay time

$$\ddot{q}_\mu + \frac{2}{\tau_\mu} \dot{q}_\mu + \Omega_\mu^2 q_\mu = \frac{\Omega_\mu^2}{c_\mu} F_\mu$$
Ponderomotive Effects

The frequency shift $\Delta \omega_\mu$ caused by the mechanical mode $\mu$ is proportional to $q_\mu$

$$\Delta \ddot{\omega}_\mu + \frac{2}{\tau_\mu} \Delta \dot{\omega}_\mu + \Omega_\mu^2 \Delta \omega_\mu = -\frac{\omega_0}{c_\mu} \left( \frac{F_\mu}{U} \right)^2 \Omega_\mu^2 U = -k_\mu \Omega_\mu^2 V^2$$

Total frequency shift: $\Delta \omega(t) = \sum_\mu \Delta \omega_\mu(t)$

Static frequency shift: $\Delta \omega_0 = \sum_\mu \Delta \omega_{0\mu} = -V^2 \sum_\mu k_\mu$

Static Lorentz coefficient: $k = \sum_\mu k_\mu$
Ponderomotive Effects – Mechanical Modes

\[ \Delta \dot{\omega}_\mu + \frac{2}{\tau_\mu} \Delta \dot{\omega}_\mu + \Omega_\mu^2 \Delta \omega_\mu = -\Omega_\mu^2 k_\mu V_0^2 + \eta(t) \]

Fluctuations around steady state:

\[ \Delta \omega_\mu = \Delta \omega_{\mu_0} + \delta \omega_\mu \]

\[ V = V_0 (1 + \delta v) \]

Linearized equation of motion for mechanical mode:

\[ \delta \ddot{\omega}_\mu + \frac{2}{\tau_\mu} \delta \dot{\omega}_\mu + \Omega_\mu^2 \delta \omega_\mu = -2\Omega_\mu^2 k_\mu V_0^2 \delta v \]

The mechanical mode is driven by fluctuations in the electromagnetic mode amplitude.

Variations in the mechanical mode amplitude causes a variation of the electromagnetic mode frequency, which can cause a variation of its amplitude.

\[ \Rightarrow \text{Closed feedback system between electromagnetic and mechanical modes, that can lead to instabilities.} \]
Lorentz Transfer Function

\[ \delta \dot{\omega}_\mu + \frac{2}{\tau_\mu} \delta \dot{\omega}_\mu + \Omega^2_\mu \delta \omega_\mu = -2 \Omega^2_\mu k_\mu V_o^2 \delta \nu \]

\[ \delta \omega_\mu(\omega) = \frac{-2 \Omega^2_\mu k_\mu V_o^2}{\left( \Omega^2_\mu - \omega^2 \right) + \frac{2}{\tau_\mu} i \omega} \delta \nu(\omega) \]

TEM-class cavities
ANL, single-spoke,
354 MHz, \( \beta=0.4 \)

simple spectrum with few modes
Lorentz Transfer Function

TM-class cavities (Jlab, 6-cell elliptical, 805 MHz, $\beta=0.61$)
Rich frequency spectrum from low to high frequencies
Large variations between cavities
GDR and SEL

Graphical representation of GDR and SEL systems. The GDR system includes a Phase Controller, an Amplitude Controller, a Gradient Detector, and a Cavity. The SEL system includes a Limiter, a Phase Controller, an Amplitude Controller, a Gradient Detector, and a Cavity.
Generator-Driven Resonator

- In a generator-driven resonator the coupling between the electromagnetic and mechanical modes can lead to two ponderomotive instabilities.

- **Monotonic instability**: Jump phenomenon where the amplitudes of the electromagnetic and mechanical modes increase or decrease exponentially until limited by non-linear effects.

- **Oscillatory instability**: The amplitudes of both modes oscillate and increase at an exponential rate until limited by non-linear effects.
Generator-Driven Resonator

Approximate stability criteria in the absence of feedback:

• Monotonic: \(-y k\mu V_o^2 < \frac{1}{2\tau}\)

• Oscillatory: \(y k\mu V_o^2 < \frac{1}{2\tau\mu} \left(1 + \frac{\tau^2 \Omega^2}{\Omega^2}\right)^2\)

where \(y = \tau (\omega_g - \omega_{co})\): normalized detuning

The monotonic instability can occur on the low frequency side when the Lorentz detuning is of the order of an electromagnetic bandwidth.
The oscillatory instability can occur on the high frequency side when the Lorentz detuning is of the order of a mechanical bandwidth.

Amplitude feedback can stabilize system with respect to ponderomotive instabilities
Self-Excited Loop

- Resonators operated in self-excited loops in the absence of feedback are free of ponderomotive instabilities. An SEL is equivalent to the ideal VCO.
  - Amplitude is stable
  - Frequency of the loop tracks the frequency of the cavity

- Phase stabilization can reintroduce instabilities, but they are easily controlled with small amount of amplitude feedback
Approximate stability criteria with phase feedback

- **Monotonic:**
  \[-y k_{\mu} V_0^2 < \frac{k_a + 1}{2\tau}\]
  
- **Oscillatory:**
  \[y k_{\mu} V_0^2 < \frac{1}{2\tau_{\mu}} \frac{(k_a + 1)^2 k_\phi}{\Omega_{\mu}^2 (k_\phi + k_a + 1)}\]

where: \(y = \tan \theta_l\), no beam loading, \(k_a, k_\phi \gg 1\), \(\tau \Omega_{\mu}\), \(\frac{\tau}{\tau_{\mu}} \ll 1\)

The stability boundary can be pushed arbitrarily far with amplitude feedback.

General stability criteria without above restrictions (including beam loading) exist.
Input-Output Variables

- Generator - driven cavity

Generator amplitude ($V_g$) \rightarrow Field amplitude ($V_o$)

Detuning ($\omega - \omega_c$) \rightarrow Cavity phase shift ($\theta_l$)

- Cavity in a self-excited loop

Limiter output ($V_g$) \rightarrow Field amplitude ($V_o$)

Loop phase shift ($\theta_l$) \rightarrow Loop frequency ($\omega$)

Ponderomotive effects

Pondermotive effects
Input-Output Variables
Generator-Driven Resonator

![Graph showing the relationship between Amplitude (Norm.) and Detuning (Norm.)](image)

![Graph showing the relationship between Caity Phase Shift and Detuning (Norm.)](image)
Input-Output Variables
Self-Excited Loop

Amplitude (Norm.)

Detuning (Norm.)

Loop Phase Shift

Loop Phase Shift
Ponderomotive Instabilities in GDR

\[ \omega_0 / 2\pi = 72 \text{ MHz}, \quad \tau = 9 \text{ msec}, \quad \Delta \omega = 18 \text{ Hz} \]

\[ \Omega_\mu / 2\pi = 89 \text{ Hz}, \quad \tau_\mu = 85 \text{ sec}, \quad k_\mu / 2\pi = 500 \text{ kHz} \]
Ponderomotive Instabilities in SEL

\[ X \frac{dY}{dt} - Y \frac{dX}{dt} = V^2 \frac{d\phi}{dt} = V^2 \delta \omega \]
Ponderomotive Instabilities in SEL

Fig. 5.5. Oscillatory stability boundary vs. amplitude feedback gain.

Fig. 5.6. Oscillatory stability boundary vs. phase feedback gain.

\[ k_{ph}^2 = 30.9 \]
\[ \tau \omega \nu \phi^2 = 550 \]
\[ \tau \nu \phi = 0.161 \]
Microphonics

• Microphonics: changes in frequency caused by connections to the external world
  — Vibrations
  — Pressure fluctuations

When phase and amplitude feedbacks are active, ponderomotive effects can change the response to external disturbances

\[
\delta \dot{\omega}_\mu + \frac{2}{\tau_\mu} \delta \omega_\mu + \Omega_\mu^2 \delta \omega_\mu = -2\Omega_\mu^2 k_\mu V_0^2 \delta v + n(t)
\]
**Microphonics**

Two extreme classes of driving terms:

- **Deterministic, monochromatic**
  - Constant, well defined frequency
  - Constant amplitude

- **Stochastic**
  - Broadband (compared to bandwidth of mechanical mode)
  - Will be modeled by gaussian stationary white noise process
Microphonics (probability density)

Single gaussian
Noise driven

Bimodal
Single-frequency driven

Multi-gaussian
Non-stationary noise

805 MHz TM

805 MHz TM

172 MHz TEM
Microphonics (frequency spectrum)

TM-class cavities (JLab, 6-cell elliptical, 805 MHz, $\beta=0.61$)
- Rich frequency spectrum from low to high frequencies
- Large variations between cavities

TEM-class cavities (ANL, single-spoke, 354 MHz, $\beta=0.4$)
- Dominated by low frequency (<10 Hz) from pressure fluctuations
- Few high frequency mechanical modes that contribute little to microphonics level.

![SNS M02, Cavity 3, Bkgnd Microphonics Spectrum, 1W](image1)

![Peak Frequency Shift (Hz)](image2)
Probability Density (histogram)

Harmonic oscillator \((\Omega_\mu, \tau_\mu)\) driven by:

- Single frequency, constant amplitude
- White noise, gaussian

\[
p(\delta \omega) = \frac{1}{\pi \sqrt{\delta \omega_{\text{max}}^2 - \delta \omega^2}}
\]

\[
p(\delta \omega) = \frac{1}{\sigma_\omega \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{\delta \omega}{\sigma_\omega} \right)^2 \right]
\]
Autocorrelation Function

\[ R_x(\tau) = \langle x(t) x(t + \tau) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t) x(t + \tau) \, dt \]

Harmonic oscillator \((\Omega_\mu, \tau_\mu)\) driven by:

Single frequency, constant amplitude

White noise, gaussian

\[ r_{\delta\omega}(\tau) = \frac{R_{\delta\omega}(\tau)}{R_{\delta\omega}(0)} = \cos(\omega_d \tau) \]

\[ r_{\delta\omega}(\tau) = \frac{R_{\delta\omega}(\tau)}{R_{\delta\omega}(0)} = \cos(\Omega_\mu \tau) e^{-|\tau/\tau_\mu|} \]
Stationary Stochastic Processes

\(x(t)\): stationary random variable

**Autocorrelation function:**

\[
R_x(\tau) = \langle x(t) x(t + \tau) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t) x(t + \tau) \, dt
\]

**Spectral Density** \(S_x(\omega)\):

Amount of power between \(\omega\) and \(d\omega\)

\(S_x(\omega)\) and \(R_x(\tau)\) are related through the Fourier Transform (Wiener-Khintchine)

\[
S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x(\tau) e^{-i\omega \tau} \, d\tau
\]

\[
R_x(\tau) = \int_{-\infty}^{\infty} S_x(\omega) e^{i\omega \tau} \, d\omega
\]

**Mean square value:**

\[
\langle x^2 \rangle = R_x(0) = \int_{-\infty}^{\infty} S_x(\omega) \, d\omega
\]
Stationary Stochastic Processes

For a stationary random process driving a linear system

\[ x(t) \xrightarrow{T(i\omega)} y(t) \]

\[ \langle y^2 \rangle = R_y(0) = \int_{-\infty}^{+\infty} S_y(\omega) \, d\omega \]
\[ \langle x^2 \rangle = R_x(0) = \int_{-\infty}^{+\infty} S_x(\omega) \, d\omega \]

\[ R_y(\tau) = R_x(\tau) : \text{auto correlation function of } y(t) \quad [x(t)] \]
\[ S_y(\omega) = S_x(\omega) : \text{spectral density of } y(t) \quad [x(t)] \]

\[ S_y(\omega) = S_x(\omega)|T(i\omega)|^2 \]

\[ \langle y^2 \rangle = \int_{-\infty}^{+\infty} S_x(\omega)|T(i\omega)|^2 \, d\omega \]
Performance of Control System

Residual phase and amplitude errors caused by microphonics
Can also be done for beam current amplitude and phase fluctuations

Assume a single mechanical oscillator of frequency $\Omega_\mu$ and decay time $\tau_\mu$
excited by white noise of spectral density $A^2$
Performance of Control System

\[
< \delta \omega_{ex}^2 > = A^2 \int_{-\infty}^{+\infty} |G_{\mu} (i\omega)|^2 d\omega = A^2 \int_{-\infty}^{+\infty} \frac{d\omega}{-\omega^2 + \frac{2}{\tau_{\mu}} i\omega + \Omega_\mu^2} = A^2 \frac{\pi \tau_{\mu}}{2\Omega_\mu^2}
\]

\[
< \delta v^2 > = A^2 \int_{-\infty}^{+\infty} |G_{\mu} (i\omega) G_{a} (i\omega)|^2 d\omega = < \delta \omega_{ex}^2 > \frac{2\Omega_{\mu}^2}{\pi \tau_{\mu}} \int_{-\infty}^{+\infty} \left| \frac{G_{a} (i\omega)}{-\omega^2 + \frac{2}{\tau_{\mu}} i\omega + \Omega_\mu^2} \right|^2 d\omega
\]

\[
< \delta \phi^2 > = A^2 \int_{-\infty}^{+\infty} |G_{\mu} (i\omega) G_{\phi} (i\omega)|^2 d\omega = < \delta \omega_{ex}^2 > \frac{2\Omega_{\mu}^2}{\pi \tau_{\mu}} \int_{-\infty}^{+\infty} \left| \frac{G_{\phi} (i\omega)}{-\omega^2 + \frac{2}{\tau_{\mu}} i\omega + \Omega_\mu^2} \right|^2 d\omega
\]
The Real World

![Probability Density](chart1)

![Probability Density](chart2)

![Microphonics](chart3)

![Normalized Autocorrelation Function](chart4)
The Real World

**Probability Density**

-10 -5 0 5 10

**Microphonics**

-10 -8 -6 -4 -2 0 2 4 6 8 10

**Normalized Autocorrelation Function**

0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

-1 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1

-10 -8 -6 -4 -2 0 2 4 6 8 10

**Time (sec)**
The Real World

Probability density

Microphonics

Normalized Autocorrelation Function
Piezo control of microphonics

MSU, 6-cell elliptical 805 MHz, $\beta=0.49$

Adaptive feedforward compensation

**Figure 2.** Active damping of helium oscillations at 2K.  
**Figure 3.** Active damping of external vibration at 2K.
Piezo Control of Microphonics

FNAL, 3-cell 3.9 GHz
Piezo Control of Microphonics

LANL 402.5 MHz scruncher cavity, $Q_L = 10^9$

Phase-locked with piezo only

Fig 3. Control for the self oscillating mode.
Final Words

- Microphonics and ponderomotive instabilities issues in high-Q SRF cavities were “hot topics” in the early days (~70s), especially in low-β applications
- They were solved and are well understood
- They are being rediscovered in medium- to high-β applications
- Today’s challenges:
  - Large scale (cavities and accelerators): need for optimization
  - Finite beam loading
    - Small but non-negligible current (e.g. RIA)
    - Low current resulting from the not quite perfect cancellation of 2 large currents (ERLs)