Computation of NRQCD Parameters in Potential Models

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Abstract

This paper gives details of computing the numerical values of the radial wave function at the origin of heavy quark-antiquark system, $|R_{n\ell}(0)|^2$, and its derivatives in potential quark models. These quantities provide numerical estimates of the non-perturbative parameters in Non-Relativistic QCD (NRQCD), an effective field theory describing decay and production of heavy quarkonium states. In particular, we computed wave functions at the origin for three different spherically symmetric potentials describing $c\bar{c}$, $b\bar{b}$ and $c\bar{b}$ bound states. The methods used for calculating the radial wave function at the origin are covered, along with the code used to produce the values given. Comparisons are made with values obtained by Eichten and Quigg [1] and Quigg and Rosner [2].

1 Introduction

NRQCD is a non-relativistic approximation to QCD for heavy quark systems. Its predictions can be systematically improved by adding higher-order terms in heavy quark velocity v and strong coupling α_s. NRQCD makes it easier to deal with the multitude of scales (m_q, $m_q v$, $m_q v^2$, ...) appearing in the calculation of production and annihilation rates of heavy quarkonium states. It allows the separation of nonrelativistic physics, formulated in terms of nonperturbative parameters, and relativistic effects absorbed in the coefficients of those parameters and written as perturbation series in α_s. These parameters cannot be computed within NRQCD but can be assigned certain power of v, then fixed by experimental data, or computed with models. A computation of those parameters in potential quark models is the main goal of this paper. We describe our method and results for $Q\bar{Q}$ states in the next three sections. Section 5 describes the ongoing effort to extend these calculations to a computation of production rates of heavy *hybrid* states. In particular, we would be able to see if the production mechanisms of the recently discovered $X(3872)$ state are consistent with its being a hybrid $c\bar{c}g$ state.

2 Quark-antiquark Systems

Heavy quark-antiquark pairs can be adequately described by the reduced radial Schrödinger equation

$$u_{n\ell}(r) \equiv rR_{n\ell}(r)$$

$$u''_{n\ell}(r) = [V_{\text{eff}}(r) - \varepsilon_{n\ell}]u_{n\ell}(r), \tag{1}$$

the effective potential

$$V_{\text{eff}}(r) \equiv 2\mu V(r) + \frac{\ell(\ell+1)}{r^2}$$

with reduced mass

$$\mu \equiv \frac{m_a m_b}{m_a + m_b}$$

composed of constituent masses m_a and m_b and scaled energy eigenvalue

$$\varepsilon_{n\ell} \equiv 2\mu E_{n\ell}.$$

2.1 Potentials and Parameters

There are a number of phenomenological potentials for modeling $c\bar{c}$, $b\bar{b}$ and $c\bar{b}$ systems. Potentials are generally constructed from the concepts of linear confinement and asymptotic freedom, which are covered in detail elsewhere [5]. Generally, parameters for the these potentials are found by fits to experimental data, allowing variation. The potentials[1] utilized in this paper are the Cornell potential,

$$V(r) = -\frac{\kappa}{r} + \frac{r}{a^2}$$

with parameters

$$m_c = 1.84\,\text{GeV}/c^2, m_b = 5.18\,\text{GeV}/c^2, \kappa = 0.52, a = 2.34\,\text{GeV}^{-1} \tag{2}$$

	$\|R_{n0}(0)\|^2$					
Level	Power Law		Logarithmic		Cornell	
$c\bar{c}$	Analytic(3)	Fit	Analytic(3)	Fit	Analytic(3)	Fit
$1S(n=0, l=0)$	0.9786	0.9787	0.7955	0.7955	1.4589	1.4584
$2S(n=1, l=0)$	0.5446	0.5446	0.4059	0.4059	0.9298	0.9297
$3S(n=2, l=0)$	0.3900	0.3900	0.2766	0.2766	0.7931	0.7929

Table 1: Values from (3) and curve fits for S levels of the $c\bar{c}$ meson

the logarithmic potential,

$$V(r) = -0.6635\text{GeV} + (0.733\text{GeV})\ln(r \times 1\,\text{GeV})$$

$$m_c = 1.5\,\text{GeV}/c^2, m_b = 4.906\,\text{GeV}/c^2$$

and the power-law potential,

$$V(r) = -8.064\,\text{GeV} + (6.898\,\text{GeV})(r \times 1\,\text{GeV})^{0.1}$$

$$m_c = 1.8\,\text{GeV}/c^2, m_b = 5.18\,\text{GeV}/c^2$$

3 Numerical Solutions

Numerical solutions to the reduced radial Schrödinger equation(1) are obtained by searching for $\varepsilon_{n\ell}$ values with which the wave function obeys the boundary conditions $u(0) \to 0$ and $u(\infty) \to 0$, following from normalization[3], and the desired n value, corresponding to the number of nodes in the radial wave function. The accompanying code written for Mathematica 5 to accomplish this task is based on the work in [3].

3.1 Wave Functions at the Origin

Values at the origin for $\ell = 0$ were obtained by the equation[4]

$$|\psi(0)|^2 = \frac{\mu}{2\pi\hbar^2}\left\langle \frac{dV(r)}{dr} \right\rangle \tag{3}$$

and

$$\psi_{n\ell m}(\vec{r}) = R_{n\ell}(r)Y_{\ell m}(\theta, \phi).$$

For $\ell > 0$ a curve fitting routine was used to estimate the function through the origin.

3.2 Origin Values for $\ell > 0$

Due to the singularity at $r = 0$ in the radial wave function, an approximation for $0 \leq u(r) \leq \delta$ is made, where δ is a small number proportional to the step size h by $\delta = h/10$ and used to account for the singularity[3]. A polynomial with the zeroth order coefficient set to 0, in accordance with the boundary conditions, is fit to the reduced radial wave equation from within the interval $\delta \leq r \leq r_{ex}$, where r_{ex} is the first extremum point. Since $R(0) = 0$ for $\ell > 0$, we find $|R^{(\ell)}(0)|^2$; first fitting $u(r)$, dividing the polynomial by r, then deriving the polynomial and setting $r = 0$. The number of data points fit is determined by the prescribed stepsize and fit interval, and the order of the polynomial is determined by the shape of the resulting wave function within that interval. A fifth degree polynomial with $h = 10^{-3}$ through the interval $10^{-4} \leq u(r) \leq 0.1$ was used in this study.

A comparison of $|R_{n0}(0)|^2$ values using equation (3) and the curve fitting technique is displayed in Table 1 to show the accuracy of the procedure.

4 Results

Table 2 contains radial values of S, P, and D excitation levels of the charmonium and bottomonium mesons under three potentials. S values are found using (3), and greater ℓ states are found with the curve fitting method. The values at the origin obtained in [1] are displayed as comparison.

4.1 Model Discrepancies

The given models should adequately predict the total mass of the systems they describe; however, the total mass of the $c\bar{c}$ meson, as determined by the Cornell potential with the previously given set of parameters (2), has a difference of 0.841 GeV with the experimentally observed value. A change of parameters[2],

$$\kappa = 0.506,\ a = 2.429\,\mathrm{GeV}^{-1}, \qquad (4)$$
$$m_c = 1.37\,\mathrm{GeV},\ m_b = 4.79\,\mathrm{GeV},$$

produces values that are closer to those found experimentally and given by the other potentials.

5 Outlook and Ongoing Work

QCD predicts the existence of the states where gluon degrees of freedom manifest, the so-called hybrid states, so it is imperative to look for the signs of those states. A recently discovered hidden-charm meson, the $X(3872)$, could be interpreted as a hybrid state. In particular, it has a mass greater then the energy threshold for production of the pair of D-mesons $D\bar{D}$, so, depending on its quantum numbers, it could decay prominently to $D\bar{D}$. This decay mode, however, has not been found. An observation can be made that this fact is consistent with the supposed properties of charm hybrid states [6], at least in the framework of Isgur-Paton flux-tube model [7] and some related models.

We compute the production rate for $X(3872)$, a recently discovered meson, in $p\bar{p}$ collisions at Tevatron and in B-decays, assuming the hybrid interpretation of this state, using the non-relativistic QCD approach to production and decays of heavy hybrid states [6]. The basic results for production rates are expressed in factorized form, separating the short-distance dynamics of $c\bar{c}$ quark production from the non-perturbative parameters describing the evolution of the $c\bar{c}$ pair into the hybrid charmonium state. As in the case of NRQCD, those parameters can be estimated in a quark model in terms of the product of the overlap integral for the glue wave function (assumed to be a number of order one) and the value of quark wave function at the origin. These values are thus necessary in determining the validity of this interpretation.

The Isug-Paton flux-tube model potential for hybrid charmonium states is

$$V(r) = -\frac{a}{r} + c + br + \frac{\pi}{r}(1 - \exp^{-fb^{1/2}r})$$

with parameters

$$a = 0.5,\ m_c = 1.77\,\mathrm{GeV/c^2},\ b = 0.18\,\mathrm{GeV}^2,\ c = -0.7\,\mathrm{GeV}^2,\ f = 1.$$

This potential is used to the calculation of the values of $c\bar{c}$ wavefunction at origin.

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	$\|R^{(\ell)}_{n\ell}(0)\|^2$						
Level	Power Law		Logarithmic		Cornell		
$c\bar{c}$	Paper	[1]	Paper	[1]	Paper	[1]	Paper (4)
$1S(n=0,l=0)$	0.978	0.999	0.795	0.815	1.458	1.454	0.784
$2S(n=1,l=0)$	0.544	0.559	0.405	0.418	0.929	0.927	0.544
$3S(n=2,l=0)$	0.390	0.410	0.276	0.286	0.793	0.791	0.476
$2P(n=0,l=1)$	0.123	0.125	0.076	0.078	0.130	0.131	0.058
$3P(n=1,l=1)$	0.126	0.131	0.073	0.076	0.185	0.186	0.085
$3D(n=0,l=2)$	0.025	0.026	0.012	0.012	0.030	0.031	0.010
$b\bar{b}$	Paper	[1]	Paper	[1]	Paper	[1]	Paper (4)
$1S(n=0,l=0)$	4.423	4.591	4.705	4.916	14.12	14.05	10.75
$2S(n=1,l=0)$	2.461	2.571	2.401	2.532	5.700	5.668	4.545
$3S(n=2,l=0)$	1.762	1.858	1.636	1.736	4.287	4.271	3.462
$2P(n=0,l=1)$	1.520	1.572	1.472	1.535	2.065	2.067	1.437
$3P(n=1,l=1)$	1.568	1.660	1.416	1.513	2.442	2.440	1.734
$3D(n=0,l=2)$	0.869	0.892	0.737	0.765	0.835	0.860	0.531
$c\bar{b}$	Paper	[1]	Paper	[1]	Paper	[1]	Paper (4)
$1S(n=0,l=0)$	1.719	1.710	1.508	1.508	3.193	3.184	1.794
$2S(n=1,l=0)$	0.957	0.950	0.769	0.770	1.769	1.764	1.091
$3S(n=2,l=0)$	0.685	0.680	0.524	0.563	1.449	1.444	0.917
$2P(n=0,l=1)$	0.352	0.327	0.220	0.239	0.342	0.342	0.163
$3P(n=1,l=1)$	0.324	0.352	0.212	0.239	0.461	0.461	0.229
$3D(n=0,l=2)$	0.095	0.101	0.051	0.055	0.098	0.102	0.392

Table 2: $|R^{(\ell)}_{n\ell}(0)|^2$ for $Q\bar{Q}$ mesons obtained with the method outlined in this paper and the values from Eichten & Quigg [1] and the change of parameters(4)