Bounds on 4D Conformal and Superconformal Field Theories

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(with David Poland [arXiv:1009.2087])

Motivation

- Near-conformal dynamics could play a role in BSM physics!
  - Walking/Conformal Technicolor [Many people...]
  - Conformal Sequestering [Luty, Sundrum '01; Schmaltz, Sundrum '06]
  - Solution to $\mu/B\mu$ problem [Roy, Schmaltz '07; Murayama, Nomura, Poland '07]
  - Flavor Hierarchies [Georgi, Nelson, Manohar '83; Nelson, Strassler '00]
  - ...

- However, many of these ideas involve statements about operator dimensions that are difficult to check.
- In non-SUSY theories, hard to calculate anything! Lattice studies may be only hope.
- In $\mathcal{N} = 1$ SCFTs, we actually know lots about chiral operators, but not much about non-chiral operators...

Example: Nelson-Strassler Flavor Models ['00]

- Idea: Matter fields T_i have large anomalous dimensions under some CFT, flavor hierarchies generated dynamically!

$$W = T_1\mathcal{O}_1 + T_2\mathcal{O}_2 + y^{ij}T_iT_jH + \dots$$

- Interactions of matter T_i with CFT operators $\mathcal{O}_i$ are marginal
- Yukawa couplings y^{ij} are irrelevant, flow to zero at a rate controlled by $\dim T_i$
- Since T_i are chiral, $\dim T_i = \frac{3}{2}R_{T_i}$
- Can write down lots of concrete models and then calculate dimensions using a-maximization! [Poland, DSD '09]

Example: Nelson-Strassler Flavor Models ['00]

- Flavor violating soft-mass operators $K \sim \frac{1}{M_{pl}^2} X^\dagger X T_i^\dagger T_j$ also flow to zero, rate depends on $\dim T_i^\dagger T_j$
- Maybe can solve SUSY flavor problem? But no 4D tools to calculate dimensions...
- Can we say *anything* about $\dim T^\dagger T$, given $\dim T$?

- Recently Rattazzi, Rychkov, Tonni, Vichi [arXiv:0807.0004, arXiv:0905.2211] addressed a similar question in non-SUSY CFTs, deriving *bounds* on $\dim \phi^2$ as a function of $\dim \phi$...

Goals

Generalizing their methods, we'll compute

- Bounds on dimensions of nonchiral operators in SCFTs
- Bounds on central charges in general CFTs and SCFTs

Outline

Outline

CFT Review: Primary Operators

- In addition to Poincaré generators, a CFT has a dilatation generator D and special conformal generators K^a
- *Primary* operators $\mathcal{O}^I(0)$ are defined by the condition

$$[K^a, \mathcal{O}^I(0)] = 0$$

  (descendants obtained by acting with P^a)
- Primary 2-pt functions $\langle \mathcal{O}^I(x_1) \mathcal{O}^J(x_2) \rangle$ and 3-pt functions $\langle \phi(x_1)\phi(x_2)\mathcal{O}^I(x_3) \rangle$ fixed by conformal symmetry in terms of dimensions and spins, up to overall coefficients $\lambda_{\mathcal{O}}$
- Higher n-pt functions *not* fixed by conformal symmetry alone, but are determined once operator spectrum and 3-pt function coefficients $\lambda_{\mathcal{O}}$ are known...

CFT Review: Operator Product Expansion

Let ϕ be a scalar primary of dimension d in a 4D CFT:

$$\phi(x)\phi(0) \quad = \sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}} C_I(x,P)\,\mathcal{O}^I(0) \qquad \text{(OPE)}$$

- Sum runs over *primary* $\mathcal{O}$'s
- $C_I(x,P)$ fixed by conformal symmetry [Dolan, Osborn '00]
- $\mathcal{O}^I = \mathcal{O}^{a_1\ldots a_l}$ can be any spin-l Lorentz representation (traceless symmetric tensor) with $l = 0, 2, \ldots$
- Unitarity tells us that $\lambda_{\mathcal{O}}$ is real

CFT Review: Conformal Block Decomposition

Use OPE to evaluate 4-point function

$$
\begin{aligned}
&\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \\
&\qquad = \sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}^2 C_I(x_{12},\partial_2) C_J(x_{34},\partial_4)\langle \mathcal{O}^I(x_2)\mathcal{O}^J(x_4)\rangle \\
&\qquad \equiv \frac{1}{x_{12}^{2d}x_{34}^{2d}} \sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}^2\, g_{\Delta,l}(u,v)
\end{aligned}
$$

- $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ conformally-invariant cross ratios.
- $g_{\Delta,l}(u,v)$ conformal block ($\Delta = \dim \mathcal{O}$ and $l =$ spin of $\mathcal{O}$)

CFT Review: Conformal Blocks

Explicit formula [Dolan, Osborn '00]

$$
\begin{aligned}
g_{\Delta,l}(u,v) &= \frac{(-1)^l}{2^l}\frac{z\bar{z}}{z-\bar{z}}[k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - z \leftrightarrow \bar{z}] \\
k_\beta(x) &= x^{\beta/2}{}_2F_1(\beta/2,\beta/2,\beta;x),
\end{aligned}
$$

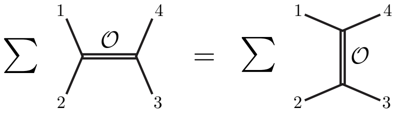
where $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$.

- Similar expressions in other even dimensions, recursion relations known in odd dimensions
- Alternatively can be viewed as eigenfunctions of the quadratic casimir of the conformal group [Dolan, Osborn '03]

CFT Review: Crossing Relations

- Four-point function $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$ is clearly symmetric under permutations of x_i
- After OPE, symmetry is non-manifest!
- Switching $x_1 \leftrightarrow x_3$ gives the "crossing relation":

$$\sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}^2 g_{\Delta,l}(u,v) \quad = \quad \left(\frac{u}{v}\right)^d \sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}^2 g_{\Delta,l}(v,u)$$



- Other permutations give no new information
- $\lambda_{\mathcal{O}}^2$ positive by unitarity

Outline

Review: Method of Rattazzi et. al. [arXiv:0807.0004]

- Let's study the OPE coefficient of a particular $\mathcal{O}_0 \in \phi \times \phi$
- We can rewrite crossing relation as

$$\underbrace{\lambda^2_{\mathcal{O}_0} F_{\Delta_0,l_0}(u,v)}_{\mathcal{O}_0} \quad = \quad \underbrace{1}_{\text{unit op.}} \quad - \quad \underbrace{\sum_{\mathcal{O}\neq\mathcal{O}_0} \lambda^2_{\mathcal{O}} F_{\Delta,l}(u,v)}_{\text{everything else}},$$

where

$$F_{\Delta,l}(u,v) \quad \equiv \quad \frac{v^d g_{\Delta,l}(u,v) - u^d g_{\Delta,l}(v,u)}{u^d - v^d}.$$

Review: Method of Rattazzi et. al. [arXiv:0807.0004]

Idea: Find a linear functional α such that

$$\begin{aligned} \alpha(F_{\Delta_0,l_0}) &= 1, \quad \text{and} \\ \alpha(F_{\Delta,l}) &\geq 0, \quad \text{for all other } \mathcal{O} \in \phi \times \phi. \end{aligned}$$

Applying to both sides:

$$\begin{aligned} \alpha\left(\lambda^2_{\mathcal{O}_0} F_{\Delta_0,l_0}\right) &= \alpha(1 - \sum_{\mathcal{O}\neq\mathcal{O}_0} \lambda^2_{\mathcal{O}} F_{\Delta,l}) \\ \lambda^2_{\mathcal{O}_0} &= \alpha(1) - \sum_{\mathcal{O}\neq\mathcal{O}_0} \lambda^2_{\mathcal{O}} \alpha(F_{\Delta,l}) \leq \alpha(1) \end{aligned}$$

since $\lambda^2_{\mathcal{O}} \geq 0$ by unitarity.

Review: Method of Rattazzi et. al. [arXiv:0807.0004]

- To make the bound $\lambda^2_{\mathcal{O}_0} \leq \alpha(1)$ as strong as possible, can *minimize* $\alpha(1)$ subject to the constraints $\alpha(F_{\Delta_0,l_0}) = 1$ and $\alpha(F_{\Delta,l}) \geq 0$ ($\mathcal{O} \neq \mathcal{O}_0$).
- This is an infinite dimensional linear programming problem... to use known algorithms (e.g., simplex) we must make it finite
- Can take α to be linear combinations of derivatives at some point in $z, \overline{z}$ space

$$\alpha : F(z,\overline{z}) \mapsto \sum_{m+n\leq 2k} a_{mn} \partial_z^m \partial_{\overline{z}}^n F(1/2, 1/2)$$

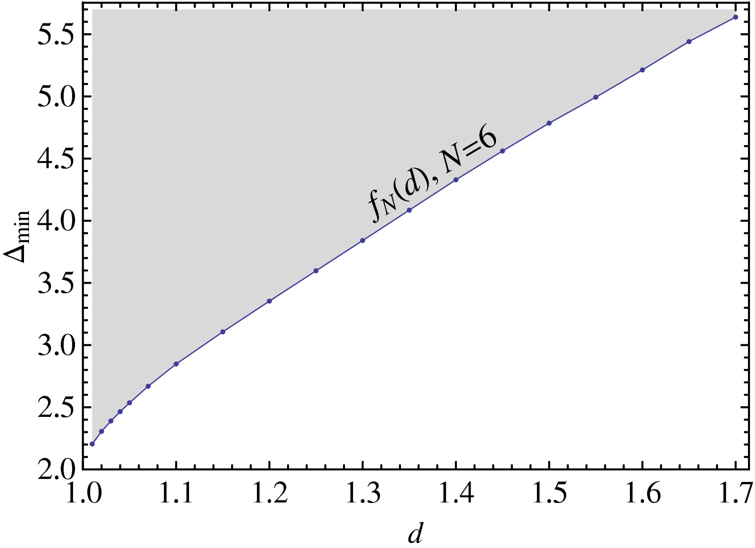
- Discretize constraints to $\alpha(F_{\Delta_i,l_i}) \geq 0$ for $D = \{(\Delta_i, l_i)\}$
- Take $k, D \to \infty$ to recover "optimal" bound

Review: Method of Rattazzi et. al. [arXiv:0807.0004]

- Can do this under any assumptions we want
- E.g., can assume that all scalars appearing in the OPE $\phi \times \phi$ have dimension larger than some $\Delta_{\text{min}} = \dim \mathcal{O}_0$
- If $\lambda^2_{\mathcal{O}_0} \leq \alpha(1) < 0$, there is a contradiction with unitarity and the assumed spectrum can be ruled out

By scanning over different Δ_{min}, one can obtain bounds on $\dim \phi^2$ as a function of $d = \dim \phi$

Bounds on $\dim \phi^2$ (taken from arXiv:0905.2211)



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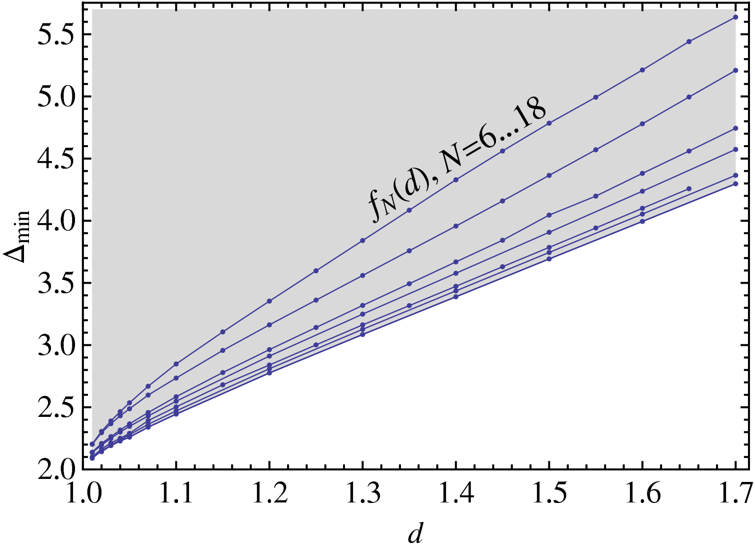

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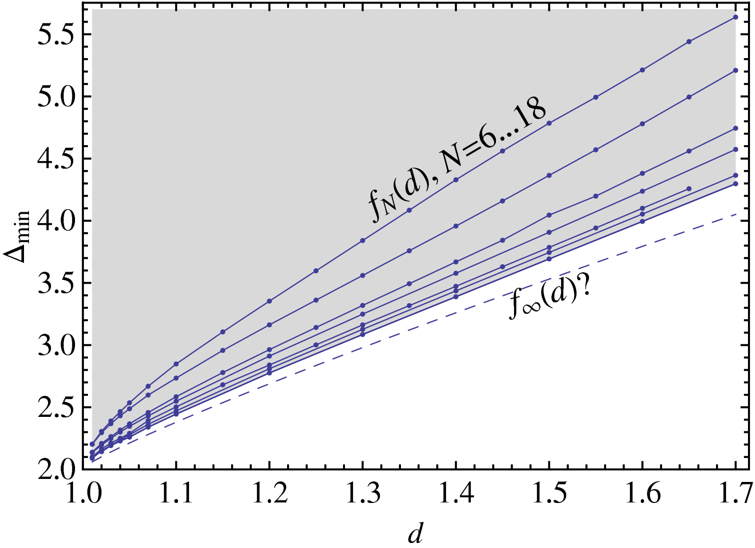

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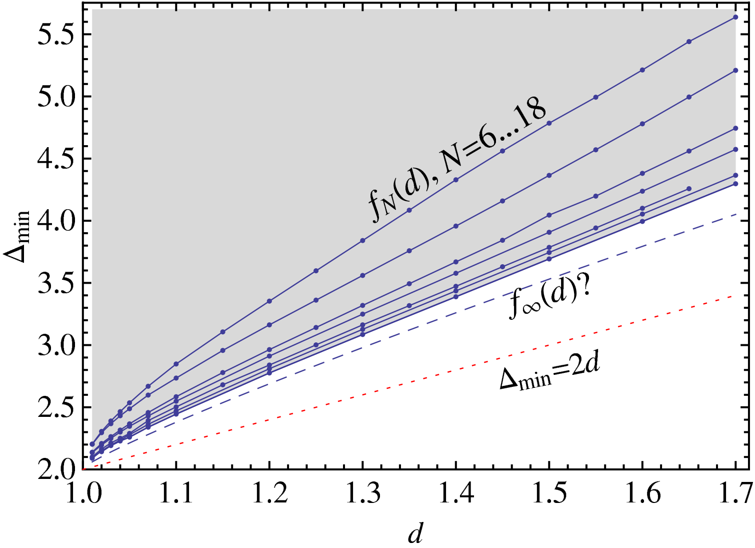


Limitations of This Method

- Only considered a single real ϕ, can't distinguish between $\mathcal{O}$'s in different global symmetry representations
- E.g., for a chiral Φ in an $\mathcal{N}=1$ SCFT, $\mathrm{Re}[\Phi]\times\mathrm{Re}[\Phi]$ contains operators from both $\Phi\times\Phi$ and $\Phi\times\Phi^\dagger$
- $\dim\Phi^2 = 2\dim\Phi$. So Φ^2 always satisfies dimension bound and we learn nothing about $\Phi^\dagger\Phi$...
- Supersymmetry also relates different conformal primaries, so we should additionally take this information into account

Let's try to generalize the method to deal with this case!

Outline

$\mathcal{N} = 1$ Superconformal Algebra

dim					
$+1$			P_a		
$+1/2$		Q_α		$\overline{Q}_{\dot\alpha}$	
0	$M_{\alpha\beta}$		D, R		$M_{\dot\alpha\dot\beta}$
$-1/2$		S_α		$\overline{S}_{\dot\alpha}$	
-1			$K_a,$		

$$\{Q, \overline{Q}\} = P \qquad\qquad \{S, \overline{S}\} = K$$

- Superconformal primary means $[S, \mathcal{O}(0)] = [\overline{S}, \mathcal{O}(0)] = 0$
- Descendents obtained by acting with $P, Q, \overline{Q}$
- Chiral means $[\overline{Q}, \phi(0)] = 0$

Superconformal Block Decomposition

ϕ: scalar chiral superconformal primary of dimension d in an SCFT (lowest component of chiral superfield Φ)

$$\langle \phi(x_1)\phi^\dagger(x_2)\phi(x_3)\phi^\dagger(x_4)\rangle = \frac{1}{x_{12}^{2d}x_{34}^{2d}} \sum_{\mathcal{O}\in\Phi\times\Phi^\dagger} |\lambda_{\mathcal{O}}|^2 (-1)^l \mathcal{G}_{\Delta,l}(u,v)$$

- Sum over superconformal primaries $\mathcal{O}^I$ with zero R-charge
- $\lambda_{\mathcal{O}}$ real for even spin $\mathcal{O}^I$, imaginary for odd spin $\mathcal{O}^I$
- $x_1 \leftrightarrow x_3$ gives crossing relation only involving $\mathcal{O}^I \in \Phi \times \Phi^\dagger$
- Must organize superconformal descendents into reps of the conformal subalgebra...

Superconformal Block Derivation

Multiplet built from $\mathcal{O}$ (generically) contains four conformal primaries with vanishing R-charge and definite spin:

name	operator	dim	spin
$\mathcal{O}$	$\mathcal{O}$	Δ	l
J, N	$Q\overline{Q}\mathcal{O} + \#P\mathcal{O}$	$\Delta + 1$	$l+1, l-1$
D	$Q^2\overline{Q}^2\mathcal{O} + \#PQ\overline{Q}\mathcal{O} + \#PP\mathcal{O}$	$\Delta + 2$	l

- Superconformal symmetry fixes coefficients of $\langle\phi\phi^\dagger J\rangle, \langle\phi\phi^\dagger N\rangle, \langle\phi\phi^\dagger D\rangle$ in terms of $\langle\phi\phi^\dagger\mathcal{O}\rangle$
- Must also normalize J, N, D to have canonical 2-pt functions
- Superconformal block is then a sum of $g_{\Delta,l}$'s for $\mathcal{O}, J, N, D$

Superconformal Block Derivation

We find,[1]

$$\mathcal{G}_{\Delta,l} = g_{\Delta,l} - \frac{(\Delta+l)}{2(\Delta+l+1)} g_{\Delta+1,l+1} - \frac{(\Delta-l-2)}{8(\Delta-l-1)} g_{\Delta+1,l-1} + \frac{(\Delta+l)(\Delta-l-2)}{16(\Delta+l+1)(\Delta-l-1)} g_{\Delta+2,l}.$$

- When unitarity bound $\Delta \geq l+2$ is saturated, multiplet is shortened.
- $\mathcal{G}_{\Delta,l}$ can also be determined from consistency with $\mathcal{N}=2$ superconformal blocks computed by Dolan and Osborn ['01].

[1] after plenty of algebra

Outline

Bounds on Dimension of $\Phi^\dagger\Phi$

Isolating the lowest dimension scalar $\Phi^\dagger\Phi \in \Phi \times \Phi^\dagger$, we have

$$|\lambda_{\Phi^\dagger\Phi}|^2 \mathcal{F}_{\Delta_{\min},0} \;=\; 1 - \sum_{\mathcal{O}\neq\Phi^\dagger\Phi} |\lambda_{\mathcal{O}}|^2 \mathcal{F}_{\Delta,l},$$

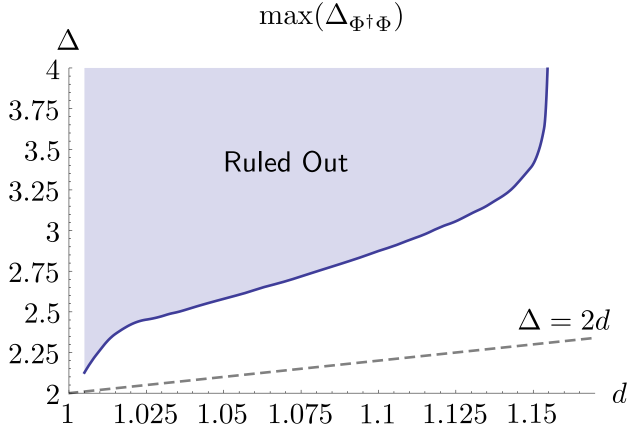
where $\Delta_{\min} = \dim \Phi^\dagger\Phi$, and $\mathcal{F}_{\Delta,l}$ is $F_{\Delta,l}$ with $g_{\Delta,l} \to (-1)^l \mathcal{G}_{\Delta,l}$.

Now minimize $\alpha(1)$ subject to

- $\alpha(\mathcal{F}_{\Delta,0}) \geq 0$ for all $\Delta \geq \Delta_{\min}$,
- $\alpha(\mathcal{F}_{\Delta,l}) \geq 0$ for all $\Delta \geq l+2$ and $l \geq 1$,
- $\alpha(\mathcal{F}_{\Delta_{\min},0}) = 1$

If $\alpha(1) < 0$, we get $|\lambda_{\Phi^\dagger\Phi}|^2 < 0 \implies \Phi^\dagger\Phi$ can't have dim $\Delta_{\min}$

Upper Bound on Dimension of $\Phi^\dagger\Phi$



- Scanning over $\Delta_{\min}$, minimizing $\alpha(1)$ over 21 dimensional space of derivatives

Flavor Currents

- If ϕ transforms under flavor symmetry with charges T^I, conserved currents J^I appear in the $\phi \times \phi^\dagger$ OPE:

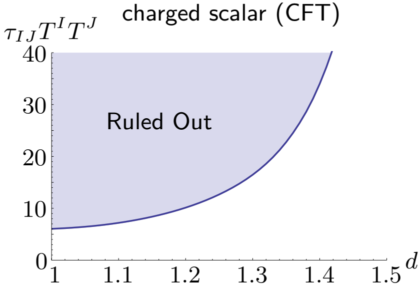
$$\langle \phi\phi^\dagger J^I \rangle \quad \sim \quad -\frac{i}{2\pi^2} T^I \qquad \text{(Ward id.)}$$

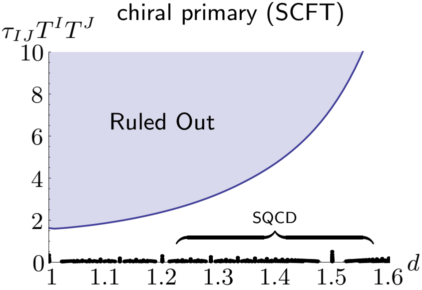
- Flavor current conformal blocks are then determined by current 2-pt functions

$$\begin{aligned} \langle J^I J^J \rangle &\sim \frac{3}{4\pi^4}\tau^{IJ} \\ \langle \phi\phi^\dagger\phi\phi^\dagger \rangle &\sim -\frac{1}{3}\tau_{IJ} T^I T^J\, g_{3,1} \qquad \text{(general CFTs)}, \\ \langle \phi\phi^\dagger\phi\phi^\dagger \rangle &\sim \quad \tau_{IJ} T^I T^J\, \mathcal{G}_{2,0} \qquad \text{(SCFTs)}, \end{aligned}$$

where $\tau_{IJ} = (\tau^{IJ})^{-1}$ (in SCFTs, $\tau^{IJ} = -3\mathrm{Tr}(R T^I T^J)$).

Upper Bounds on $\tau_{IJ}T^IT^J$





- Example: SUSY QCD with $\frac{3}{2}N_c < N_f < 3N_c$, consider $\langle MM^\dagger MM^\dagger\rangle$. $d = 3 - \frac{3N_c}{N_f}$ and $\tau_{IJ}T^IT^J = \frac{2}{3}\frac{N_f-1}{N_c^2}$
- For a $U(1)$ in an SCFT, $\frac{Q^2}{-3\sum_i(R_i-1)Q_i^2}$ can't be too big.
- In dual AdS$_5$, $(8\pi^2 L)\tau_{IJ} = g_{IJ}^2$. Gauge coupling can't be too strong in presence of charged scalar.

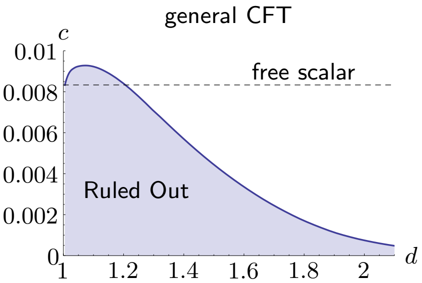
The Stress Tensor

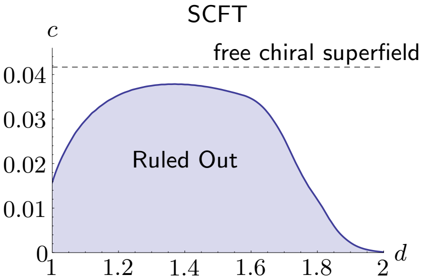
- Ward identity ensures $T^{ab} \in \phi \times \phi$
- $\langle TT \rangle$ is proportional to the central charge c (trace anomaly $16\pi^2 \langle T^a_a \rangle = c(\mathrm{Weyl})^2 - a(\mathrm{Euler})$)
- In an SCFT, T lives in the supercurrent multiplet $\mathcal{J}^a = J^a_R + \theta\sigma_b\overline{\theta}T^{ab} + \ldots$, and c is determined in terms of $U(1)_R$ anomalies
- Conformal block contributions are

$$\langle \phi\phi\phi\phi \rangle \sim \frac{d^2}{90c}\, g_{4,2} \qquad \text{(general CFTs)}$$

$$\langle \phi\phi^\dagger\phi\phi^\dagger \rangle \sim -\frac{d^2}{36c}\, \mathcal{G}_{3,1} \qquad \text{(SCFTs)}$$

Lower Bound on c in General CFT





- In dual AdS$_5$, $c \sim \pi^2 L^3 M_P^3$. Gravity can't be too strong in presence of bulk scalar.

Outline

Outlook

We calculated:

- Superconformal blocks
- Bound $\dim \Phi^\dagger\Phi < f_{\Phi^\dagger\Phi}(d)$
- Bound $\tau_{IJ}T^IT^J \leq f_\tau(d)$ in CFT, SCFT
- Bound $c \geq f_c(d)$ in CFT, SCFT

In the future, we'd like:

- Stronger bounds to make contact with BSM motivation! Improved numerics and better algorithms.
- SUSY theories that come close to saturating bounds on τ, c.
- Bounds in other numbers of dimensions.
- Understand bounds from bulk dual perspective.