M5-branes wrapped on Riemann surfaces

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Plan

- Motivations - results
- N=2 theories from wrapped M5-branes
- N=1 theories and gravity dual
- Anomaly polynomial and central charges
- 3d theories
- Conclusions - future directions

Based on:  
FB, Benvenuti, Tachikawa  0906.0359  
FB, Tachikawa, Wecht  0909.1327  
Alday, FB, Tachikawa  0909.4776
Motivations

We study the effective 4d theories from M5-branes wrapped on Riemann surfaces.

• Understanding the M5-brane theory

  1 M5: self-dual $B_{\mu\nu}, \phi^i=1\ldots5, \psi^a=1,2$

  $N$ M5: 6d $\mathcal{N}=(2,0)$ SCFT with $SO(5)_R$
  No Lagrangian
Motivations

- **S-duality**

Four examples:

- **N=4** SU($N$) SYM: SL(2,$Z$) S-duality

- **N=2** SU(2) SQCD with 4 flavors 
  SL(2,$Z$) S-duality

- **N=2** SU(3) SQCD with 6 flavors 
  $\Gamma^0(2)$ S-duality + Argyres-Seiberg duality to 
  SU(2) w/ 1 flavor, coupled to $E_6$ SCFT

- **N=1** SU($N$) SQCD w/ 2$N$ flavors + quartic $W$ Seiberg duality
Motivations

• New isolated SCFT without Lagrangian: generalize $E_{6,7,8}$ SCFT's of Minahan-Nemeschansky

• BPS quantities exactly computable via 2d - 4d correspondence with Liouville/Toda AGT relations

Alday Gaiotto Tachikawa
M5-branes on $\Sigma$ with N=2

- $N$ M5-branes wrapped on a Riemann surface $\Sigma_{g,n}$ with $n$ punctures, with N=2 twist ($\rightarrow$ the normal bundle is $T^*\Sigma$)

$\rightarrow$ N=2 SCFTs whose diagram “reproduces” the surface $\Sigma$

- A Riemann surface with punctures admits pant decompositions in terms of tubes and triskelions:

\[ \text{Diagram images of tube and triskelion decompositions} \]

Gaiotto
M5-branes on $\Sigma$ with N=2

• For each decomposition, consider a limit (for the complex structure) with very long tubes

• Long tube $\rightarrow$ weakly coupled gauge group
  The triskelion is not weakly coupled: isolated SCFT

• What is the puncture?
  - defect of 6d theory (some field diverges)
  - transverse M5-branes $\rightarrow$ flavors, global symmetries
  - alternative IIB description...

• A triskelion is an interacting SCFT (without coupling) with $G_1 \times G_2 \times G_3$ global symmetry

• Tubes gauge global symmetries together
Triskelions

- 4 hyper $\rightarrow$ SU(2)$\times$SU(2)$\times$SU(2)
- $N^2$ hyper $\rightarrow$ SU($N$)$\times$U(1)$\times$SU($N$)
- E$_6$ theory $\rightarrow$ E$_6 \supset$ SU(3)$\times$SU(3)$\times$SU(3)
Sicilian theories

- all SU(2) quivers with bi- or tri-fundamental matter (L)
- all Hanany-Witten constructions in IIA with D4/NS5 → linear and elliptic quivers (L)
e.g.: SU(N) SQCD with 2N flavors, conifold theory, ...
- all superconformal linear quivers of SU groups (L)
- E\textsubscript{6,7,8} of Minahan-Nemeschenski and higher rank (non-L)
- theories on pure M5s (non-L) → dual to Maldacena-Nunez gravity solutions
- generalization to SO/Sp gauge groups
- much more...
M5-branes on $\Sigma$ with $N=2$

- The complex structure moduli space of $\Sigma_{g,n}$ is

$$M_{g,n} = \widetilde{M}_{g,n}/\Gamma$$

$$\Gamma = \pi_1(M_{g,n})$$

Teichmüller/(large diffeo x permutations)

It equals the parameter space of marginal couplings of the 4d theory ($\Gamma$ being S-duality of couplings)

- Different pant decompositions give dual descriptions, à la Argyres-Seiberg
Argyres-Seiberg duality

- Argyres-Seiberg duality in N=2 SU(3) SQCD w/ $N_f = 6$

in terms of Minahan-Nemeshansky $E_6$ theory.
The Coulomb branch

- N=2 supersymmetry: $T^*\Sigma$ hyperKahler ($\frac{1}{2}$ SUSY) with M5-branes wrapping a meromorphic curve ($\frac{1}{2}$ SUSY) of degree $N$:

\[
x_N = \phi_N(z) + \phi_{N-1}(z)x + \ldots + \phi_2(z)x^{N-2}
\]

$\phi_j(z)$: degree-$j$ differentials,
poles of order up to $j-1$ at the punctures

Eq: **Seiberg-Witten curve**. SW differential: $\lambda_{SW} = x$

- **Coulomb branch**: moduli space of multi-differentials with allowed poles. Riemann-Roch:

\[
\text{moduli of } \phi_j = (2j-1)(g-1) + \sum_{p=1}^{n} d_{p,j}
\]
$T_N$ theory

- Theory on $N$ M5-branes wrapped on $S^2$ with 3 “maximal” punctures
- $N=2$ isolated SCFT, with $SU(N)^3$ flavor symmetry
- Coulomb branch parametrized by dimension-$k$ operators
  \[ u_k^{(i)} \quad \text{for} \quad k = 3 \ldots N, \ i = 1 \ldots k-2 \quad \text{flavor singlets} \]
- Higgs branch (?) contains three dimension-2 operators
  \[ \mu_{i=1,2,3} \quad \text{adjoint of} \ SU(N)_i \quad \text{flavor} \]

Examples:
- $T_2$ - 4 free hypers $Q_{\alpha\beta\gamma}$
- $T_3$ - $E_6$ theory of Minahan-Nemeshansky
Maldacena-Nunez + max punctures

- Building blocks: $T_N$ theory

- Glue (gauge) maximal punctures together ($\rightarrow$ SCFT)

- Generate Sicilian theories: $T_{g,n}$
Flavor symmetry at the puncture

FB, Benvenuti, Tachikawa

• Understand classification of pole structures and associated flavor symmetry from IIB construction.
  Focus on $T_N$ theory:
  $(p,q)$ 5-brane webs compactified on $S^1$ realize the $T_N$ theory

• Generalization: end multiple 5-branes on the same 7-brane

• Punctures classified by partitions of $N$
  $\leftrightarrow$ Young diagrams with $N$ boxes
Flavor symmetry at the puncture

• Each stack of $n_k$ identical objects
  $\rightarrow U(n_k)$ symmetry:

$$S\left[ \prod_{k \geq 1} U(n_k) \right]$$

• Non-maximal punctures as effective theories along the Higgs branch of the maximal puncture
M5-branes on $\Sigma$ with N=1

- Worldvolume point of view: wrap the branes with N=1 twist
  \[ \text{SO}(5)_R \rightarrow \text{SO}(4) \cong \text{SU}(2) \times \text{SU}(2)_F \]
  \[ \downarrow \]
  \[ \text{U}(1)_R \]
  embed the spin connection $U(1)_\Sigma$ into $U(1)_R$

- We expect a new infinite family of N=1 SCFTs, with intricate net of S-dualities

- Parameters: complex structure moduli space of $\Sigma$ with $\text{SU}(2)$ flat bundle
Massive deformed $\mathbb{T}_{n,g}$

- Field theory point of view: deform $N=2$ to $N=1$ with mass for the adjoint scalars in vector multiplets

- *Closed class under S-duality:*
  - only gauge groups provide dimension-2 operators $\text{Tr} \Phi_s^2$

- $N=2$ SUSY requires

- Mass deformation

- The theory flows to a fixed point with "quartic" superpotential

\[
W = \sum_s \text{Tr} \Phi_s (\mu_{a,i} + \mu_{b,j})
\]

\[
\delta W = \sum_s m_s \text{Tr} \Phi_s^2
\]

\[
\rightarrow \quad W = \sum_s \frac{1}{m_s} \text{Tr} (\mu_{a,i} + \mu_{b,j})^2
\]
NSVZ formula

- The all-loop beta-function is computed by the NSVZ formula. It depends on the representation and anomalous dimension of fundamental fields.

  → what with non-Lagrangian sector?

\[ \beta_{8\pi^2} = 3 T[adj] - \sum_i T[r_i](1 - \gamma_i) + K \]

\[ K \delta^{ab} = 3 \text{Tr} \ R_{N=1} T^a T^b \]

-'t Hooft anomaly

- The low-energy N=1 R-symmetry is a combination of \( U(1)_R \times SU(2)_R \)

\[ R_{IR} = \frac{1}{2} R_{N=2} + I_3 \]

Flavor current central charge \( k_G \delta^{ab} = -2 \text{Tr} \ R_{N=2} T^a T^b \)

\( T_N \) and sons: \( k_G = 2N \)
Conformal manifold

- Compute the dimension of the conformal manifold (exactly marginal deformations) à la Leigh-Strassler:
  - write down all marginal operators, e.g. $\text{Tr} \, \mu^2, \text{Tr} \, \mu \mu, \text{Tr} \, W_\alpha W^\alpha$
  - count real relations from vanishing beta-functions
  - count phases removed by field redefinitions
    (Resembles a Kahler quotient...)

- No punctures: $\dim_{\mathbb{C}} M_C = 6(g-1)$

Maximal punctures: $\dim_{\mathbb{C}} M_C = 6(g-1) + 2n_N$
Gravity dual

- Maldacena-Nunez solution with N=1 twist:
  \[ w[AdS_5] \times w[H^2] \times \tilde{S}^3 \times I \]
  squashed \( S^3 \) preserves \( U(1)_R \times SU(2)_F \), fibered over \( H^2 \)

- To get compact Riemann surface, mod out by Fuchsian group:
  \[ C = H^2 / \Gamma \quad \Gamma \subset SL(2,\mathbb{R}) \]

- \( SU(2)_F \) "unwanted" symmetry
  Introduce \( SU(2)_F \) Wilson lines (N=1) on \( C \):
  \[ \tilde{S}^3 \to E \to H^2 \quad E_C = E / \Gamma_W \quad \Gamma_W \subset SL(2,\mathbb{R}) \times SU(2)_F \]

- Moduli space of \( \Sigma_g \) with \( SU(2) \) Wilson lines:
  \[ 6(g - 1) = \dim M_C \]
Gravity dual with maximal punctures

- The gravity dual (Gaiotto Maldacena) of the maximal puncture is a $\mathbb{Z}_N$ orbifold singularity

\[ \left( z, x_1, x_2 \right) \rightarrow \left( e^{\frac{2\pi i}{N}} z, e^{-\frac{2\pi i}{N}} x_1, x_2 \right) \]

- The N=1 twist alone would give the action

\[ \left( z, x_1, x_2 \right) \rightarrow \left( e^{\frac{2\pi i}{N}} z, e^{-\frac{\pi i}{N}} x_1, e^{-\frac{\pi i}{N}} x_2 \right) \]

→ the SU(2)$_F$ monodromy has fixed conjugacy class:

\[ \left( x_1, x_2 \right) \rightarrow \left( e^{-\frac{2\pi i}{N}} x_1, e^{\frac{2\pi i}{N}} x_2 \right) \]

- Moduli space of $\Sigma_{g,n}$ with SU(2) Wilson lines and constrained monodromies:

\[ 6(g - 1) + 2n_N = \dim M_C \]
Central charges from SUGRA

- Central charges computed by AdS$_5$ radius and curvature corrections

- Packaged in the anomaly polynomial of the 6d theory:

\[ I_8[A_{N-1}] = \frac{N-1}{48} \left[ p_2(N) - p_2(T) + \frac{1}{4} \left( p_1(N) - p_1(T) \right)^2 \right] + \frac{N^3 - N}{24} p_2(N) \]

- Twist embedding of U(1)$_R$ bundle $F$ into the normal bundle $N$, impose N=1 SUSY and integrate on $\Sigma$

- Compare with the anomaly polynomial of the 4d theory:

\[ I_6 = \frac{\text{Tr} R^3}{6} c_1(F)^3 + \frac{\text{Tr} R}{24} c_1(F) p_1(T_4) \]

Witten; Harvey Minasian Moore
Central charges from SUGRA

- Get: \( \text{Tr } R = N (g - 1) \) , \( \text{Tr } R^3 = N^3 (g - 1) \)

- Exploiting SUSY, reproduce the central charges:

  \[
  a = \frac{3}{32} \left[ 3 \text{Tr } R^3 - \text{Tr } R \right] \\
c = \frac{1}{32} \left[ 9 \text{Tr } R^3 - 5 \text{Tr } R \right]
  \]

\( \rightarrow \) matching with field theory

\[
  a = (g - 1) \frac{9 N^3 - 3 N - 6}{32} \\
c = (g - 1) \frac{9 N^3 - 5 N - 4}{32}
  \]
2d – 4d correspondence

- The Nekrasov partition function of the 4d theory obtained wrapping M5s on $\Sigma_{g,n}$ with equivariant deformations $\varepsilon_1, \varepsilon_2$ is equal to the conformal blocks of Liouville/Toda theory on $\Sigma_{g,n}$ with parameter $b^2 = \varepsilon_1 / \varepsilon_2$

- More observables: Wilson – 't Hooft loops, surface operators, …

- Hard to understand the relation between $\varepsilon_{1,2}$ of $\Omega$-background and central charge of Liouville/Toda (quantum property)
Toda central charge from 6d anomaly

Alday FB Tachikawa

• Compute the 2d central charge compactifying the anomaly polynomial. Compactify the 6d $N = (2,0)$ theory on $\Sigma \times X_4$.

Exploit $N = (0,2)$ of the 2d theory on $\Sigma$

• 6d anomaly polynomial for ADE series without center of mass:

$$I_8[G] = \frac{r_G}{48} \left[ p_2(N) - p_2(T) + \frac{1}{4} \left( p_1(N) - p_1(T) \right)^2 \right] + \frac{d_G h_G}{24} p_2(N)$$

Intriligator; Yi

• First step: embed the spin connection of $X_4$ in $SO(5)_R$.

$\rightarrow N = (0,2)$ SUSY in 2d

• Integrate on $X_4$: dependence on $\chi(X_4)$ and $P_1(X_4)$
• Compare with 2d anomaly polynomial, and use N=(0,2) SUSY:

\[ I_4 = \frac{c_R}{6} c_1(F)^2 + \frac{c_L - c_R}{24} p_1(T_2) \]

We get the central charges of the 2d theory on Σ.

• Nekrasov's partition function is an equivariant integral on \( X_4 = \mathbb{R}^4 \) with equivariant parameters \( \varepsilon_{1,2} \) with respect to a \( U(1)^2 \) action. Integrated equivariant classes (use localization formula):

\[ P_1(\mathbb{R}^4) = \int \varepsilon_1^2 + \varepsilon_2^2 = \frac{\varepsilon_1^2 + \varepsilon_2^2}{\varepsilon_1 \varepsilon_2} \quad \chi(\mathbb{R}^4) = \int \varepsilon_1 \varepsilon_2 = 1 \]

• Second step: embed the spin connection of Σ into \( U(1)_R \) → topological twisting of right sector

\[ b^2 = \varepsilon_1 / \varepsilon_2 \quad c_R \to 0 \quad c_L = r_G + \left( b + \frac{1}{b} \right)^2 d_G h_G \]
3d Sicilian theories

- Compactify N=2 4d theories on $S^1$
  3d Sicilian theories flow to N=4 SCFT fixed point

  Coulomb branch & Higgs branch, both hyperKahler
  - Coulomb: $T^{2n}$ fibration over 4d Coulomb branch
  - Higgs: same as 4d Higgs branch

- M5 on $S^1 \times \Sigma \rightarrow$ D4 (5d SYM) on $\Sigma$

  3d Coulomb (finite $R$): Hitchin moduli space on $\Sigma$ (with punctures)

  \[
  F - \phi \wedge \phi = 0 \\
  D\phi = D^* \phi = 0
  \]

- The mirror: quiver theory!

- Hitchin moduli space around fixed point is a quiver variety: Higgs branch of mirror quiver.

FB Tachikawa Xie, work in progress

Gaiotto Moore Neitzke
Future directions

- Higgs branch (of $T_N$)
- more on N=1: chiral theories
  compactify some 6d N=(1,0) theories
- 2d – 4d correspondence: extension to N=1
  surface operators, domain walls, ...
- 3d Sicilian theories, CS couplings,
  mirror symmetry, SUGRA solutions...