Introduction to Modern methods on Scattering amplitude (tree)

color decomposition planer loop, "dual conformal symmetry" hep-th/0907.5418

hep-th/1008.2958

YM amplitude I, II, III, VI ⟷ SYM (N=4) I, II, III, V, VI, VII

IV ↕ simplest QFT ↑ (String theory) I, III, V

Gravity amplitude II, III, VI SUGRA (N=8) II, III, V, VI

Why should we study this today?

1. Phenomenalogical reason LHC, high point Jet
2. theoritically reason: a. What's the simplest QFT? (six paritcle...) scalar, YM, N=4 SYM (recusive relation, Loop structure)
   N=4 ⟶ b. N=8 SUGRA finiteness (Quantum Gravity)
   c. Non-perturbative scattering amplitude?

Why should we go beyond Feymann rules?

Feymann rules gives everything (perturbatively)

disadvantage

1. too many diagrams (each is NOT gauge independent)
2. each diagram contains too many terms
3. For loop, it is very complicated

"Modern" method

I color decomposition II spin helicity formulism

III recursive relation (BCFW)

KK relation, BCJ relation

IV String theory (perturbative) method

Monodromy, KLT

V On-shell Supersymmetry (coherent states)

BCFW for coherent state

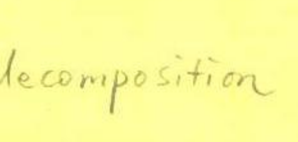
VI twistor space VII AdS/CFT

I Color decomposition [hep-ph/9601359 Lance Dixon] [Polchinski]

YM amplitude (all incoming) M-point

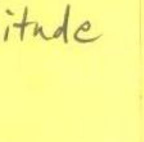
(k_1, ϵ_1, a_1) (k_M, ϵ_M, a_M)

$k_i^2 = 0$, $k_i \cdot \epsilon_i = 0$, $\sum_i k_i = 0$

$A_M^{tree} = \sum$  NO cyclic order of the legs!

physical Ward id. a_i k_i

but color factor X kinematic factor mixed up in every vertice!



$= g f^{abc} [g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu]$

⇓ final result

a long string of contracted $f^{abe} f^{cde} \ldots\ldots$

Extract the color part first!

$A^{tree}_{M\text{ gluon}} = g^{M-2} \sum_{\sigma \in S^{M-1}} tr(t^{a_{\sigma(1)}} t^{a_{\sigma(2)}} \cdots t^{a_{\sigma(M)}}) A(\{k_{\sigma(1)}, \epsilon_{\sigma(1)}\}, \{k_{\sigma(2)}, \epsilon_{\sigma(2)}\} \ldots)$

(M-1)! terms

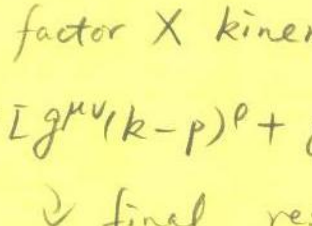
partial amplitude ordered

No color!!!

$\sigma(M) \equiv M$

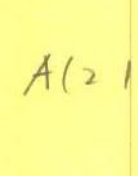
$A^{partial}$ is calculated by "color-free", ordered diagram

$A(2\,1\,3\,4)$

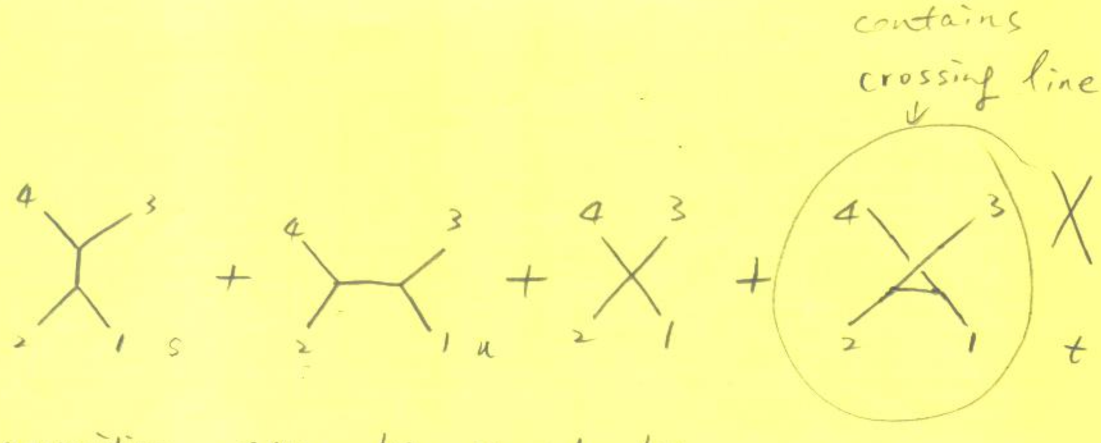


fix the cyclic order
NO CROSSING LINE!

normalization!
no color!



$= \frac{i}{\sqrt{2}} [g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu]$

$A(2134) =$ 

Color decomposition can be proved by

① $f^{abc} \propto tr((t^a t^b t^c) - tr(t^b t^a t^c))$. . .
$f^{abc} f^{cde} \propto tr([t^a, t^b][t^d, t^e]) = tr(t^a t^b t^d t^e) - tr(t^b t^a t^d t^e)$. . .

② t'Hooft double line

③ open string with chan-paton factor

$$A^{tree,\, string}_{M} = \sum_{\substack{\sigma \in S^{M-1} \\ (M-1)!\ \text{term}}} tr(t^{a_{\sigma(1)}} t^{a_{\sigma(2)}} \ldots) A^{string}(\{k_{\sigma(1)}, \epsilon_{\sigma(1)}\}, \{k_{\sigma(2)}, \epsilon_{\sigma(2)}\} \ldots)$$

$i \to j$

$|k, \epsilon, a\rangle = t^a_{ij} |k, \epsilon, ij\rangle$ ← chan-paton factor

$\alpha' \to 0$, low energy limit

$$A^{string,\, partial} \to A^{partial,\, YM}$$

Comment: (1) $A^{partial}$ is gauge-invariant ($A^{partial}(\{k_i, \epsilon_i\}) = A^{partial}(\{k_i, \epsilon_i + c_i k_i\})$)

it is easy to prove it by string theory

Gluon vertex $\epsilon_\mu \partial^\mu X e^{ik\cdot X} \to \ldots + k_\mu \partial^\mu X e^{ik\cdot X}$ ← total derivative

(2) If $G = SU(N)$, N large, it is easy to get total color-summed cross section

$$\sum_{a_1 a_2 a_3 a_4} tr(t^{a_1} t^{a_2} t^{a_3} t^{a_4}) (tr(t^{a_1} t^{a_2} t^{a_3} t^{a_4}))^*$$

"same order" dominates

$$\sum_{a_1 a_2 a_3 a_4} tr(t^{a_1} t^{a_2} t^{a_3} t^{a_4}) (tr(t^{a_2} t^{a_1} t^{a_3} t^{a_4}))^* \sim \frac{1}{N^2}$$

(3) Other color decomposition possible based on Heterotic string [Tye, Zhang hep-th/1003.1732]

$(M-3)!$ terms

$$A_4 \sim t\left(\frac{C_s}{s} - \frac{C_t}{t}\right) A^{partial}(1, 3, 2, 4)$$

$C_s = f^{abc} f^{cde} \qquad C_t = f^{adc} f^{cbe}$

II spinor helicity formalism [hep-ph/9601359 L. Dixon] [hep-th/0504194 F. Cachazo P. Svrček]

still Feynman rules

still two many invariants, $\epsilon_i \cdot \epsilon_j$, $k_i \cdot \epsilon_j$, $\epsilon^{\mu\nu\rho\sigma} k^i_\mu k^j_\nu k^k_\rho k^l_\sigma$

(k_i^μ, a, h_i) Consider partial amplitude with definite helicities

$$A(1^+, 2^+, 3^- \ldots\ldots)$$

$h_i = \pm$

What's the role of ϵ_i^μ?

Write vectors as product of spinors.

$$(p_\mu \sigma^\mu)_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} \qquad \sigma^\mu = (\sigma^0, \sigma^1, \sigma^2, \sigma^3)$$

λ_a: $-\frac{1}{2}$, $\tilde{\lambda}_{\dot{a}}$: $+\frac{1}{2}$ Weyl spinors, not Grassmanian!

$$\lambda^* = +\tilde{\lambda} \qquad \begin{pmatrix} \lambda_a \\ 0 \end{pmatrix} -\frac{1}{2} \qquad \begin{pmatrix} 0 \\ \lambda^{\dot{a}} \end{pmatrix} +\frac{1}{2} \qquad \lambda^{\dot{a}} = \epsilon^{\dot{a}\dot{b}} \lambda_{\dot{b}}$$

$\lambda_a \to e^{i\theta/2} \lambda_a \quad \tilde{\lambda}_{\dot{a}} \to e^{-i\theta/2} \tilde{\lambda}_{\dot{a}}$ does not change p_μ

rotation θ along momenta direction

General Lorentz transformation, see [Peskin]

$$\langle \lambda \lambda' \rangle = \epsilon_{ab} \lambda^a \lambda^b, \quad [\tilde{\lambda} \tilde{\lambda}'] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}^{\dot{a}} \tilde{\lambda}'^{\dot{b}}$$

"angled" "squared" Lorentz invariant!

$$\langle \lambda \lambda \rangle = 0 \qquad [\tilde{\lambda} \tilde{\lambda}] = 0$$

$$p_\mu p'^\mu = \frac{1}{2} p_{a\dot{a}} p'^{\dot{a}a} = \frac{1}{2} \langle \lambda \lambda' \rangle [\tilde{\lambda} \tilde{\lambda}']$$

$$\epsilon^+_{a\dot{a}} = \frac{\mu_a \tilde{\lambda}_{\dot{a}}}{\langle \mu, \lambda \rangle}, \quad \epsilon^-_{a\dot{a}} = \frac{\lambda_a \tilde{\mu}_{\dot{a}}}{[\tilde{\lambda}, \tilde{\mu}]} \quad (\mu \text{ is arbitary})$$

$$\epsilon^+_\mu \cdot p^\mu = \frac{1}{2} [\tilde{\lambda} \tilde{\lambda}] = 0 \qquad \epsilon^-_\mu \cdot p^\mu = \frac{1}{2} \langle \lambda \lambda \rangle = 0$$

$$\epsilon^+_\mu \to e^{i\theta} \epsilon^+_\mu \qquad \epsilon^-_\mu \to e^{-i\theta} \epsilon^-_\mu$$

$$\epsilon^+_\mu \cdot \epsilon^{\mu -} = -1$$

✓

YM $A(1^+, 2^+, \ldots i^-, \ldots j^-, \ldots M^+)/\langle ij\rangle^4$

$= A(1^+, 2^+, \ldots k^-, \ldots l^-, \ldots M^+)/\langle kl\rangle^4$ not an accident!

$\Rightarrow$ super Ward identity

tree level YM is effectively supersymmetry

How does this simplify the result?

$A(1^- 2^- 3^+ 4^+) = ?$ $\quad \dfrac{(\epsilon\cdot k)(\epsilon\cdot k)(\epsilon\cdot\epsilon)}{k^2} + (\epsilon\cdot\epsilon)(\epsilon\cdot\epsilon)$ by dimension anlysis

Polarization 1: $\dfrac{1\tilde{3}}{[13]}$ 2: $\dfrac{2\tilde{3}}{[23]}$ 3: $\dfrac{1\tilde{3}}{\langle 13\rangle}$ 4: $\dfrac{2\tilde{3}}{\langle 23\rangle}$

	ϵ_1^-	ϵ_2^-	ϵ_3^+	ϵ_4^+
ϵ_1^-	0			
ϵ_2^-	0	0		
ϵ_3^+	0	0	0	
ϵ_4^+	0	$\frac{1}{2}\frac{\langle 34\rangle[21]}{\langle 32\rangle[41]}$	0	0

	k_1	k_2	k_3	k_4
ϵ_1^-	0		0	
ϵ_2^-		0	0	
ϵ_3^+	0		0	
ϵ_4^+	0			0

$A(1^- 2^- 3^+ 4^+) =$ 

$= \dfrac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$

little group rotation by $1 \to e^{-i\theta/2} 1$ $\quad \tilde{1} \to e^{+i\theta/2}\tilde{1}$ $\quad A(1^+ \ldots) \to e^{i\theta} A(1^+ \ldots)$

Similarry $A(1^- 2^+ 3^+ 4^+) = 0$ ([HW], choose a particle ϵ_2^+, such that all $(\epsilon\cdot\epsilon)$ vanish)

MHV amplitude, In general, YM theory helicity is not conserved $\sum h_i \neq 0$ for an amplitude

but,

$A(1^+ 2^+ \ldots M^+)$ all plus, vanish (need the assistance of SUSY)

$A(1^- 2^+ \ldots M^+)$ one minus, rest plus vanish

MHV $A(1^+ 2^+, \ldots i^- j^- M^+)$ two minus, rest plus Non-vanishing

$= \dfrac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle \cdots \langle M-1, M\rangle\langle M1\rangle}$ Parke-Taylor

NMHV

(Cannot be proved by Spinor helicity & Feynman diagram)

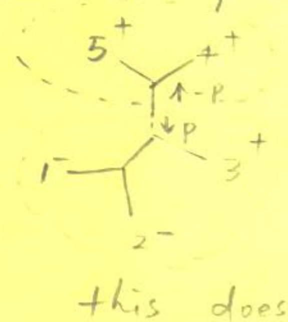
[R. Britto, F. Cachazo
B. Feng, E. Witten
hep-th/0501052]

III recursive relation (BCFW)

$A(1^- 2^- 3^+ 4^+ 5^+)$ and even high point

Is it possible to construct higher point amplitude

Naively,



$$A(1^- 2^- 3^+ 4^+ 5^+) = \frac{1}{k_4 \cdot k_5} A(1^-, 2^-, 3^+, P)\, A(-P, 4^+, 5^+)$$
+ other cuts

X

this doesn't help! $P^2 \neq 0$ $A(1^- 2^- 3^+ P)$ not a gluon amplitude

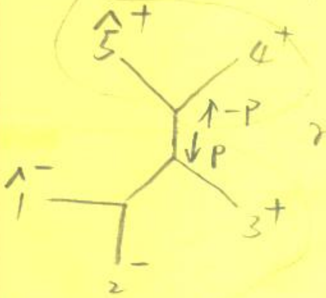
Make $P^2 = 0$!

$\hat{P}_i = P_i - z q \quad \hat{P}_j = P_j + z q \qquad q \cdot P_i = 0 \quad q \cdot P_j = 0 \quad q^2 = 0$

$q = i \tilde{j}$ complex

$(\hat{\tilde{i}} = \tilde{i} - z \tilde{j}\ ,\ \hat{j} = j + z i)$

required $P^2 = (p_4 + \hat{p}_5)^2 = 0$



BCFW

$$A(1^- 2^- 3^+ 4^+ 5^+) = \sum_h \frac{1}{P^2} A(\hat{1}^- 2^- 3^+ \hat{P}^{+h})\, A(-\hat{P}^{-h}, 4^+, \hat{5}^+)$$
+ other cut ✓

$$A(1, \dots i \dots j \dots M) = \sum_{\text{cuts}, h} \frac{1}{P^2} A_L(\dots \hat{i} \dots \hat{P}^h)\, A_R(\hat{P}^{-h} \dots \hat{j} \dots)$$
real momentum   complex momentum

Proof: $A(z) = A(\dots p_i - z q, \ p_j + z q, \dots)$

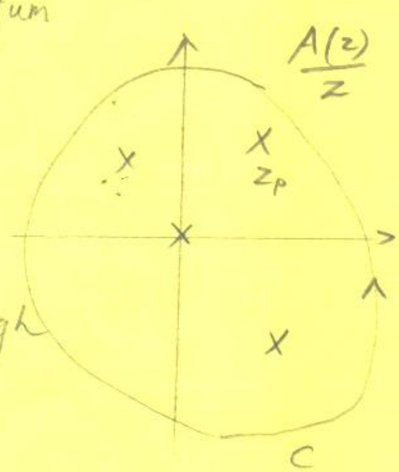
based on Feynman diagram → [BCFW] $A(z) \sim \frac{1}{z}$ when $z \to \infty$

[N. Arkani-Hamed
J. Kaplan
hep-th/0801.2385]

large z enhanced "spin-Lorentz" symmetry

$$\oint_C \frac{A(z)}{z} dz = 0$$

C large enough



pick up the residue $\frac{A_L(z_P) A_R(z_P)}{P^2}$

$$A(0) = -\frac{1}{z_P} \lim_{z \to z_P} A(z)(z - z_P) + \dots$$

$$\hat{P}^2(z_P) = 0 \Rightarrow (P + z_P q)^2 = 0 \Rightarrow P^2 + 2 z_P P \cdot q = 0 \Rightarrow z_P = -\frac{1}{2} \cdot \frac{P^2}{P \cdot q}$$

$$A(z) \cdot (z - z_P) = A_L(z_P) A_R(z_P) \cdot \frac{z - z_P}{P^2 + 2 z P \cdot q} = \frac{(z - z_P)}{2 P \cdot q (z_P - z)} \cdot A_L(z_P) A_R(z_P)$$

Continue the computation

$\hat{P}_1 = P_1 - zq$, $\hat{P}_5 = P_5 + zq$ $\qquad q = 1\tilde{5}$

$$A(1^- 2^- 3^+ 4^+ 5^+) = \frac{1}{(p_4+p_5)^2} A(\hat{1}^- 2^- 3^+ \hat{P}^+)\, A(-\hat{P}^-, 4^+, \hat{5}^+) + \text{other cuts}$$

determine $\hat{P} = p_4 + p_5 + zq$

$(p_4 + p_5 + zq)^2 = 0 \qquad 4\tilde{4} + (5 + 1\, z_p)\tilde{5} = \lambda_p \tilde{\lambda}_p$

make $5 + 1 \cdot z_p = c_p 4 \qquad \langle 15\rangle = c_p \langle 14\rangle \qquad \langle 45\rangle + \langle 41\rangle z_p = 0$

$$z_p = \frac{\langle 45\rangle}{\langle 14\rangle}$$

$\lambda_p = 4 \qquad \tilde{\lambda}_p = \left(\tilde{4} + \frac{\langle 15\rangle}{\langle 14\rangle}\tilde{5}\right)$

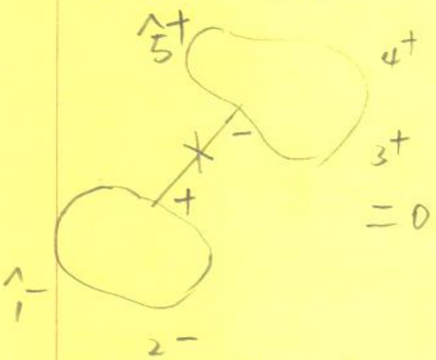
$1 \to 1 \qquad \tilde{1} \to \tilde{1} - \frac{\langle 45\rangle}{\langle 14\rangle}\cdot\tilde{5}$

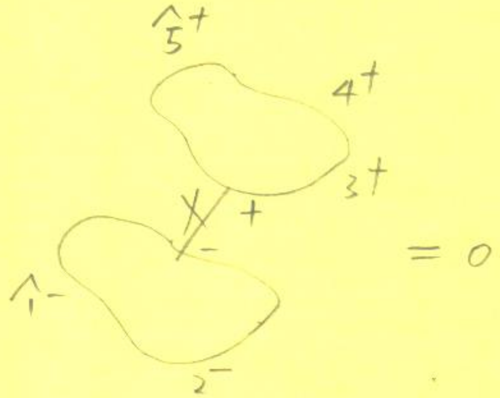
$$A(\hat{1}^- 2^- 3^+ \hat{P}^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$$A(-\hat{P}^- 4^+ 5^+) = \frac{[45]^4}{\frac{\langle 15\rangle}{\langle 14\rangle}[45][45][45]} = \frac{\langle 14\rangle[45]}{\langle 15\rangle}$$

$(p_4 + p_5)^2 = \langle 45\rangle[45]$

Other cut





$$A(1^- 2^- 3^+ 4^+ 5^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

by induction, Parke-Taylor formula proved

[HW] Check that BCFW does NOT work for $\lambda\phi^4$ theory!
($A(z) \to$ constant, $z \to \infty$) 4pt - 6pt

3-pt amplitude $\Rightarrow$ all tree amplitude $\Rightarrow$ 1 loop
BCFW

$\mathcal{N}=4$ SYM ✓

Gravity ✓

$\mathcal{N}=8$ SUGRA ✓

$\Big\}\ A^{grav}(z) \sim \frac{1}{z^2}$

bonus identity $\oint_C \overset{grav}{A}(z) = 0$

superstring theory ✓ [C. Cheung, D. O'Connell, B. Wecht hep-th/1002.4674]

Other kind of recursive relation:

KK relations

$$A(12345) + A(21345) + A(13245) + A(13425) = 0$$

$$(M-1)! \to (M-2)!$$

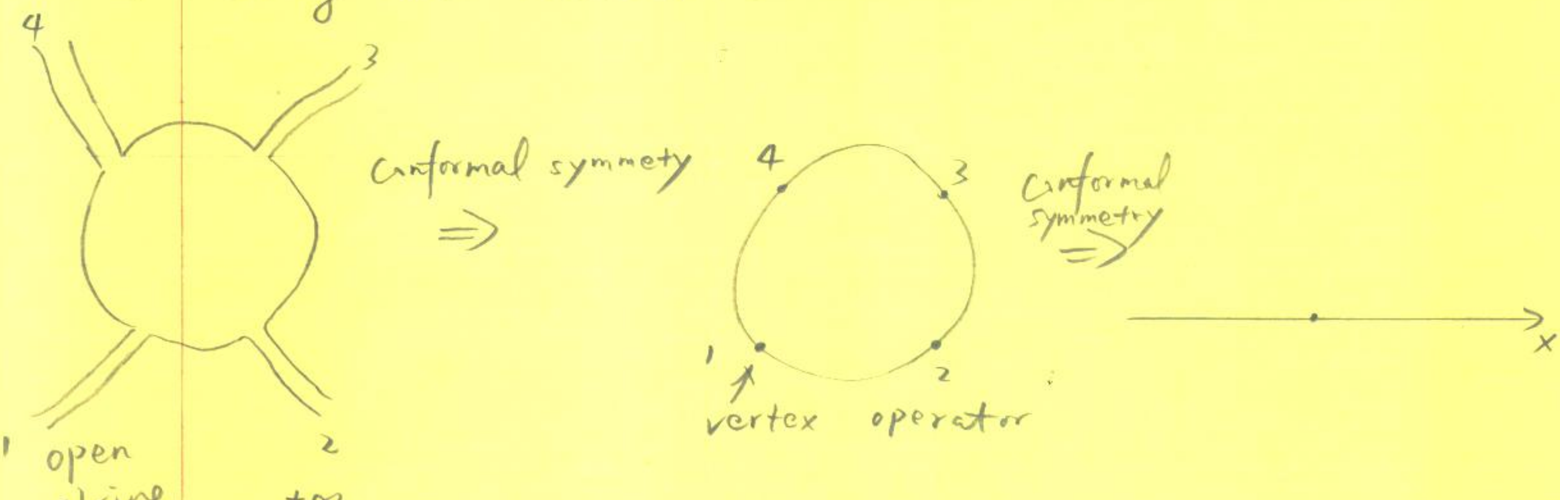
BCJ relations

$$-S_{12}A(21345) + S_{23}A(13245) + (S_{23}+S_{24})A(13425) = 0$$

$$(M-2)! \to (M-3)!$$

Can be proved by BCFW [N. Bohr, P. Damgaard B. Feng, T. Sondergaard hep-th/1003.4367]

IV String-base methods



$$\int_{-\infty}^{+\infty} dx_i\, \epsilon_{i\mu}\, \partial X_i^{\mu}(x)\, \underset{\downarrow\ \mathcal{O}(x_i)}{e^{ik_iX_i}}$$ for each incoming particle (gluon)

$$A^{\text{string}}_{\text{partial}} \propto \left(\prod_{i=1}^{M}\int_{-\infty}^{+\infty} dx_i\right)\theta(x_2-x_1)\,\theta(x_3-x_2)\ldots\theta(x_M-x_{M-1})\ (\text{Fadder-Popov determinant})$$

$$\cdot \langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\ldots\mathcal{O}(x_M)\rangle$$ → contains α', regge slope

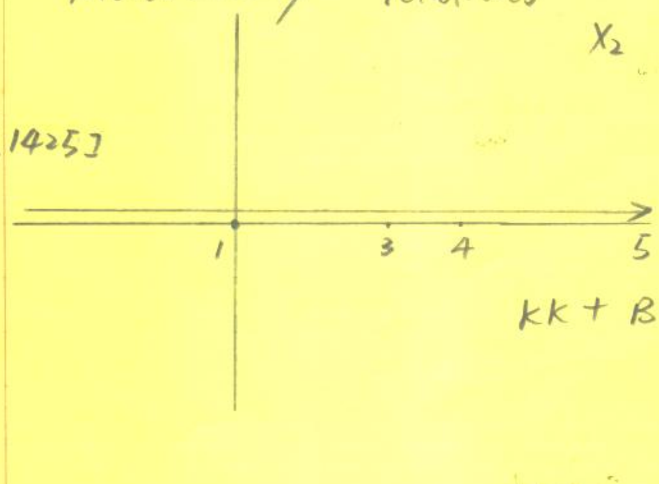
For M-gluon tree amplitude, (M-3)-integrals needed

$\alpha' \to 0$ or $k_i k_j \to 0$ $\forall i,j$ $\quad A^{\text{string}}_{\text{partial}} \to A^{\text{YM}}_{\text{partial}}$

[N. Bohr, P. Damgaard, P. Vanhove, hep-th/0907.1425]

[hep-th/1003.1732]

Monodromy relations



$$e^{-\frac{\alpha'}{2}k_1\cdot k_2}A(21345) + A(12345) + e^{\frac{\alpha'}{2}k_2\cdot k_3}A(13245) + e^{\frac{\alpha'}{2}(k_3+k_4)\cdot k_2}A(13425) = 0$$

$\alpha' \to 0$

KK + BCJ relations

Einstein theory

$$A(1^{--}2^{--}3^{++}4^{++}) = \langle 14\rangle[14]\cdot\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\cdot\frac{[34]^4}{[12][23][34][41]}$$

$$= \frac{\langle 12\rangle^4[34]^4}{-t\,\langle 12\rangle[13]\langle 34\rangle[24]} = \frac{\langle 12\rangle^4[34]^4}{stu}$$

KLT relations

$$|A(1^{--}2^{--}3^{++}4^{++})| = \frac{(t+u)^3}{tu} = \frac{t^2}{u} - 3s + \frac{u^2}{t}$$

"Einstein amplitude" (hard)

[N. Kan
K. Kobayashi
T. Hanada
K. Shiraishi
hep-th/0902.0412
]

perturbative field

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,R \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

derive Feynman rules (with gauge fixing terms)

non-renormalizable! infinite # of vertices

[H. KAWAI
D.C. LEWELLAN
H. TYE]
Nucl. phys. B269 (1986) 1-23

$\pm 2 \qquad\qquad \pm 1 \qquad \pm 1$

KLT relation: Graviton = Gluon × Gluon

$$\int d^2z\,\epsilon^{++}_{\mu\nu}\,\partial X^\mu(z)\,\bar\partial X^\nu(\bar z)\,e^{ik\cdot X(z,\bar z)} \qquad X(z,\bar z) = X_L(z) + X_R(\bar z)$$

$$\epsilon^{+}_{\mu}\,\partial X^\mu(z)\,e^{ik\cdot X_L(z)} \quad \times \quad \epsilon^{+}_{\nu}\,\bar\partial X^\nu(\bar z)\,e^{ik\cdot X_R(z)}$$

How to realize $z, \bar z$ as TWO independent?

3-pt $\quad A^{\text{graviton, string}}(1,2,3) = A^{\text{gluon, string}}(1,2,3)\,A^{\text{gluon, string}}(1,2,3)$

4-pt $\quad A^{\text{graviton, string}}(1,2,3,4) \neq A^{\text{gluon, string}}(1,2,3,4)\,A^{\text{gluon, string}}(1,2,3,4)$

double pole ??? X $\qquad$ pole $\qquad$ pole

by careful contour rotations

$$A^{\text{graviton, string}}(1,2,3,4) = \sin\left(\frac{\pi}{2}\alpha' t\right) A^{\text{gluon, string}}(1,2,3,4)\,A^{\text{gluon, string}}(1,3,2,4)$$

$\alpha'\to 0$ $\qquad$ no order for 1, 2, 3, 4 $\qquad$ ordered for 1, 2, 3, 4 $\qquad$ $\alpha'\to 0$

Einstein amplitude $\qquad \longleftrightarrow$ magic is guaranteed by BCJ identities $\qquad$ YM amplitude

All tree amplitude KLT generated.

$A(1,2,3,4,5) =$

Gravity MHV formula

[N. Arkani-Hamed, F. Cachazo, J. Kaplan, hep-th 0808.1446]

On-shell Supersymmetry $N=4$ SYM, $N=8$ SUGRA

tree amplitude

	$N=4$	$N=8$	
helicity	$+1$	$+2$	
	$+\frac{1}{2}$ $+\frac{1}{2}$ $+\frac{1}{2}$ $+\frac{1}{2}$	$+\frac{3}{2}$	$\times 8$
	0 0 0 0 0 0	1	$\times 28$
	$-\frac{1}{2}$ $-\frac{1}{2}$ $-\frac{1}{2}$ $-\frac{1}{2}$	$+\frac{1}{2}$	$\times 56$
	-1	0	$\times 70$
		$-\frac{1}{2}$	$\times 56$
		-1	$\times 28$
		$-\frac{3}{2}$	$\times 8$
		-2	$\times 1$

Unified notation $N=2S$, $S=1,2$

$Q_{I\alpha}$, $\bar{Q}^{I\dot{\alpha}}$ $I=1,\ldots,2S$

$Q_{I\alpha}$: $SU(4S)_R$, $\square$ $\bar{Q}^{I\dot{\alpha}}$: $SU(4S)_R$, $\bar{\square}$

$$Q_{\alpha I}|-s\rangle = \lambda_\alpha|-s+\tfrac{1}{2}\rangle_I, \quad \bar{Q}^{\dot{\alpha}I}|+s\rangle = \tilde{\lambda}^{\dot{\alpha}}|+s-\tfrac{1}{2}\rangle^I$$

particles labeled by $|IJK\ldots, \lambda, \tilde{\lambda}\rangle$

antisymmetric $SU(4S)_R$ reps.

Continuous states, (coherent)

only use 1 for each particle

$$|\bar{\eta},\lambda,\tilde{\lambda}\rangle = e^{\bar{Q}^{I\dot{\alpha}}\bar{w}_{\dot{\alpha}}\bar{\eta}_I}|+s,\lambda,\tilde{\lambda}\rangle \qquad [\bar{w},\tilde{\lambda}]=1$$

$$|\eta,\lambda,\tilde{\lambda}\rangle = e^{Q_{I\alpha}w^\alpha\eta^I}|-s,\lambda,\tilde{\lambda}\rangle \qquad \langle w,\lambda\rangle = 1$$

$\eta=0$ $+s$ particle $\bar{\eta}=0$ $-s$ particle

$\bar{\eta}$, η are Grassmanian, λ, $\tilde{\lambda}$ are c-number spinors!

$$e^{Q_{I\alpha}\xi^{I\alpha}}|\eta\rangle = |\eta+\langle\xi\lambda\rangle\rangle, \quad e^{Q_{I\alpha}\xi^{I\alpha}}|\bar{\eta}\rangle = \pm e^{\bar{\eta}_I\langle\lambda\xi^I\rangle}|\bar{\eta}\rangle$$

Q shift $|\eta\rangle$ but diagonalizes in $|\bar{\eta}\rangle$ states.

$$|\bar{\eta}\rangle = \int d^N\eta\, e^{\eta\bar{\eta}}|\eta\rangle \qquad |\eta\rangle = \int d^N\bar{\eta}\, e^{\bar{\eta}\eta}|\bar{\eta}\rangle \text{ (not trivial!)}$$

The amplitude

$$A(\{\eta_i,\lambda_i,\tilde{\lambda}_i\},\{\bar{\eta}_{\bar{i}},\lambda_{\bar{i}},\tilde{\lambda}_{\bar{i}}\}) \quad SU(N)_R \text{ invariant}$$

It's better to use both η_i and $\bar{\eta}_{\bar{i}}$, because it's easier to construct $SU(N)_R$ invariants like $\eta_{iI}\bar{\eta}_{\bar{i}}^I$

Supersymmetry in amplitude

by Q: "super distance" ζ: $A(\eta_i;\tilde{\eta}_{\tilde{i}}) = e^{\sum_j \langle\lambda_j\zeta\rangle\tilde{\eta}_{\tilde{j}}} A(\eta_i + \langle\lambda_i\zeta\rangle, \tilde{\eta}_{\tilde{i}})$

by $\bar{Q}$: "super distance" $\bar{\zeta}$: $A(\eta_i;\tilde{\eta}_{\tilde{i}}) = e^{\sum_j [\tilde{\lambda}_j\bar{\zeta}]\eta_j} A(\eta_i, \tilde{\eta}_{\tilde{i}} + [\tilde{\lambda}_{\tilde{i}}\bar{\zeta}])$

η_{iI} N-component $\qquad \zeta_I^{\ a}$ 2N-component

$\tilde{\eta}_{\tilde{i}}^{\ I}$ N-component $\qquad \bar{\zeta}^{I\dot{a}}$ 2N-component

It is possible to set 2 η_i & 2 $\tilde{\eta}_{\tilde{i}}^{\ I}$ to be zero

Example: 4-pt amplitude in $N=4$ SYM

$$A(\eta_1, \eta_2, \tilde{\eta}_3, \tilde{\eta}_4)$$

set $\eta_1, \eta_2 = 0 \qquad \eta_1 + \langle\lambda_1\zeta\rangle = 0 \qquad \eta_2 + \langle\lambda_2\zeta\rangle = 0$

Q $\qquad \Rightarrow \quad \zeta_{I\alpha} = (\eta_{2I}\lambda_{1\alpha} - \eta_{1I}\lambda_{2\alpha})/\langle 12\rangle$

$\langle\lambda_3\zeta\rangle = \langle 31\rangle/\langle 12\rangle\,\eta_2 - \langle 32\rangle/\langle 12\rangle\,\eta_1$

$\langle\lambda_4\zeta\rangle = \langle 41\rangle/\langle 12\rangle\,\eta_2 - \langle 42\rangle/\langle 12\rangle\,\eta_1$

$N=4$ SYM $A(\text{gluon})$ is the same as YM!

$$A(\eta_1, \eta_2, \tilde{\eta}_3, \tilde{\eta}_4) = \exp\left((\eta_1\ \eta_2)\begin{pmatrix} \frac{\langle 23\rangle}{\langle 12\rangle} & \frac{\langle 24\rangle}{\langle 12\rangle} \\ \frac{\langle 31\rangle}{\langle 12\rangle} & \frac{\langle 41\rangle}{\langle 12\rangle} \end{pmatrix}\begin{pmatrix} \tilde{\eta}_3 \\ \tilde{\eta}_4 \end{pmatrix}\right) A(0, 0, \tilde{\eta}_3, \tilde{\eta}_4)$$

$$= \exp\left(\qquad\right) A(0, 0, \tilde{0}; \tilde{0}) = \exp\left(\qquad\right) \underset{\displaystyle\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}}{\underset{\shortparallel}{A(1^- 2^- 3^+ 4^+)}}$$

Comment:

(1) $SU(4)_R$ invariant $\qquad \eta_i\tilde{\eta}_{\tilde{i}}$

(2) expand η to get different amplitude

$$A(g^-, \underset{I\neq J}{\varphi^{IJ}}, \underset{K\neq L}{\varphi_{KL}}, g^+) = \exp\left(\frac{\langle 31\rangle}{\langle 12\rangle}\eta_2\cdot\tilde{\eta}_3\right)\Big|_{\eta_2^I\eta_2^J\,\eta_3^K\eta_3^L} A(1^- 2^- 3^+ 4^+)$$

$SU(4)_R$ symmetry

$$= (\delta^I{}_K \delta^J{}_L + \delta^I{}_L \delta^J{}_K) \frac{\langle 31\rangle^2}{\langle 12\rangle^2} \cdot \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

HW $\quad A(g^-, g^-, \varphi_{IJ}, g^+) = 0 \qquad SU(4)_R$ violating

Higher point, BCFW (SUPER VERSION)

$$M(\{\eta_1(0), \lambda_1(0), \tilde\lambda_1\}, \{\eta_2, \lambda_2, \tilde\lambda_2(0)\}, \eta_i)$$

$$= \sum_{\text{cuts}} \int d^N\eta \; M_L(\{\eta_1(z_P), \lambda_1(z_P), \tilde\lambda_1\}, \eta, \eta_L) \frac{1}{P^2} M_R(\{\eta_2, \lambda_2, \tilde\lambda_2(z_P)\}, \eta, \eta_R)$$

$$\eta_1(z) = \eta_1 + z\eta_2$$