Evaluation of a New Cavity Focusing Theory

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Outline

1. Transport matrices
2. Cavity focusing
3. Initial project goals
4. What happened?
5. Plans for the future
Transport Matrices

Introduction

\[
\begin{pmatrix}
    r \\
    r'
\end{pmatrix}_f = \begin{pmatrix}
    \text{transport matrix}
\end{pmatrix}
\begin{pmatrix}
    r \\
    r'
\end{pmatrix}_i
\]

\[
r' = \frac{dr}{dz} = \frac{v_r}{v_z} \approx \theta
\]

CAVITY

initial \quad \text{transport matrix} \quad \text{final}
Transport Matrices
Accelerator elements in series

\[
\begin{pmatrix}
  r \\
  r'
\end{pmatrix}_f =
\begin{pmatrix}
  \text{magnet matrix} \\
  \text{drift matrix}
\end{pmatrix}
\begin{pmatrix}
  \text{cavity matrix} \\
  \text{drift matrix}
\end{pmatrix}
\begin{pmatrix}
  r \\
  r'
\end{pmatrix}_i
\]
Cavity Focusing
What does the radial force look like?

The particle experiences an oscillatory radial force inside the cavity.

\[ F_r = q(E_r - v_z B_\phi) \]

radial force on particle with
- constant \( r \)
- \( v_z = c \)

Net force = 0

But actual particles have changing \( r \)…
The radial force is stronger farther from the axis.

How does this affect the particle motion?
Cavity Focusing
r-dependence of the radial force

The radial force is stronger farther from the axis.

Graph showing the radial force increase with radial position.

Average force is inward $\implies$ Focusing!

How does this affect the particle motion?

Stronger inward force, weaker outward force.
Cavity Focusing
r-dependence of the radial force

The radial force is stronger farther from the axis.

![Graph showing the radial force as a function of radial position. The graph suggests a non-linear relationship.]
The radial force is stronger farther from the axis.

Linear?

Yes, for small enough $r$. 
The Beginning

Initial project goals

I started with

• a paper by J. Rosenzweig and L. Serafini
  in which they derive a transport matrix

• an alternate matrix theory developed by G. Hoffstaetter

I set out to

• compare the two matrices to each other

• compare the results of both matrix theories to actual particle motion
For the past several weeks, I have

- read the paper by Rosenzweig and Serafini
- learned how both derivations go
- written programs in \textit{Mathematica} to
  - read in data about the cavity fields
  - numerically solve equations of motion
  - plot actual and matrix trajectories
- discovered needed adjustments and additions to Georg’s theory
The Rosenzweig and Serafini matrix:

$$\begin{pmatrix} \cos \left( \sqrt{\frac{\eta}{8}} \ln \frac{\gamma_f}{\gamma_i} \right) & \frac{\cos(\Delta \phi)}{\sqrt{\eta/8}} \frac{\gamma_i}{\gamma_f} \sin \left( \sqrt{\frac{\eta}{8}} \ln \frac{\gamma_f}{\gamma_i} \right) \\ -\frac{\sqrt{\eta/8}}{\cos(\Delta \phi)} \frac{\gamma'}{\gamma_f} \sin \left( \sqrt{\frac{\eta}{8}} \ln \frac{\gamma_f}{\gamma_i} \right) & \frac{\gamma_i}{\gamma_f} \cos \left( \sqrt{\frac{\eta}{8}} \ln \frac{\gamma_f}{\gamma_i} \right) \end{pmatrix}$$

The Hoffstaetter matrix:

$$\begin{pmatrix} \cos \left( \epsilon \ln \frac{p_f}{p_i} \right) & \frac{1}{\epsilon} \frac{p_i}{p_f} \sin \left( \epsilon \ln \frac{p_f}{p_i} \right) \\ -\epsilon \frac{p'}{p_f} \sin \left( \epsilon \ln \frac{p_f}{p_i} \right) & \frac{p_i}{p_f} \cos \left( \epsilon \ln \frac{p_f}{p_i} \right) \end{pmatrix}$$
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\[
\begin{pmatrix}
\cos \left( \frac{\sqrt{\eta/8}}{\cos(\Delta \phi)} \ln \frac{\gamma_f}{\gamma_i} \right) & \frac{\cos(\Delta \phi)}{\sqrt{\eta/8}} \frac{\gamma_i}{\gamma_f} \sin \left( \frac{\sqrt{\eta/8}}{\cos(\Delta \phi)} \ln \frac{\gamma_f}{\gamma_i} \right) \\
- \frac{\sqrt{\eta/8}}{\cos(\Delta \phi)} \frac{\gamma'}{\gamma_f} \sin \left( \frac{\sqrt{\eta/8}}{\cos(\Delta \phi)} \ln \frac{\gamma_f}{\gamma_i} \right) & \frac{\gamma_i}{\gamma_f} \cos \left( \frac{\sqrt{\eta/8}}{\cos(\Delta \phi)} \ln \frac{\gamma_f}{\gamma_i} \right)
\end{pmatrix}
\]

The Hoffstaetter matrix:

\[
\begin{pmatrix}
\cos \left( \epsilon \ln \frac{p_f}{p_i} \right) & \frac{1}{\epsilon} \frac{p_i}{p_f'} \sin \left( \epsilon \ln \frac{p_f}{p_i} \right) \\
- \epsilon \frac{p'}{p_f} \sin \left( \epsilon \ln \frac{p_f}{p_i} \right) & \frac{p_i}{p_f} \cos \left( \epsilon \ln \frac{p_f}{p_i} \right)
\end{pmatrix}
\]

Are they the same?

- Does \( \epsilon = \frac{\sqrt{\eta/8}}{\cos(\Delta \phi)} \) ?
- Does \( p = \gamma \) ?
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\[
\begin{pmatrix}
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Are they the same?

- Does $\epsilon = \frac{\sqrt{\eta/8}}{\cos(\Delta \phi)}$? Yes
- Does $p = \gamma$?
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-\frac{\sqrt{\eta/8}}{\cos(\Delta\phi)} \frac{\gamma'}{\gamma_f} \sin\left(\frac{\sqrt{\eta/8}}{\cos(\Delta\phi)} \ln \frac{\gamma_f}{\gamma_i}\right) & \frac{\gamma_i}{\gamma_f} \cos\left(\frac{\sqrt{\eta/8}}{\cos(\Delta\phi)} \ln \frac{\gamma_f}{\gamma_i}\right)
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\end{pmatrix}
\]

Are they the same?

- Does \( \epsilon = \frac{\sqrt{\eta/8}}{\cos(\Delta\phi)} \)? Yes
- Does \( p = \gamma \)? No, but they are proportional \( p = \gamma m \nu \)
The Future
What comes next?

That’s not the end of the story!
It is the central matrices that are the same, but you also need edge matrices for the cavity entrance and exit

- Georg’s theory did not originally include this
- we don’t have it fully worked out yet

Possibilities for the future

- find better way to deal with the cavity entrance/exit
- see where the high $\gamma$ approximation breaks down
  - improve by including $\frac{1}{\gamma^2}$ terms?
- check with ERL injector parameters
  - and compare to real-life measurements
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