How to generate random numbers with a power (or any other) density*

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We derive the algorithm to generate a one-dimensional random number distribution with a power density, based on the uniform random number generator in $[0,1]$. For completeness, the algorithm for the case of the most general one-dimensional density is also derived and used to derive a list of algorithms for several other distributions of practical use.

This note was written for practical convenience only; the methods and the results are well known.

I. BASIC CASE: UNIFORM DENSITY.

Suppose that we want to generate a set of n random numbers $\{x\}$ such that the distribution of the x's, in the limit $n \to \infty$, is uniform in $x_1 \le x \le x_2$ and vanishes outside this interval (here x_1 and x_2 are given, with $x_1 < x_2$). In the limit $n \to \infty$, the distribution of the x's is therefore

$$\frac{dN}{dx} = \begin{cases} k & \text{if } x_1 \le x \le x_2 \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

where $k > 0$ is a constant.

The algorithm to generate the x's is, clearly,

$$x = \alpha\hat{u} + \beta \tag{2}$$

where $\hat{u}$ denotes a uniform random number in $[0,1]$, and where the constants α and β are determined by the requirements that $x = x_1$ when $\hat{u} = 0$ and $x = x_2$ when $\hat{u} = 1$, respectively, hence

$$x = (x_2 - x_1)\hat{u} + x_1 \tag{3}$$

Note that the normalization constant

$$K \equiv \int_{x_1}^{x_2} dx\, \frac{dN}{dx} = k(x_2 - x_1) \tag{4}$$

plays no direct role in the algorithm: only the ratio $K/k = x_2 - x_1$ matters, and the only requirement on it is that it must be finite.

II. POWER DENSITY.

In this case we want to generate random numbers x such that their distribution, in the $n \to \infty$ limit, is

$$\frac{dN}{dx} = kx^p \tag{5}$$

in the region $x_1 \le x \le x_2$, for a specified value of p.

While p can, in general, be an arbitrary real number, we first impose the constraints $p \ne -1$ and $0 < x_1 < x_2 < \infty$. Removal of these constraints is possible in some cases, as discussed below. The power density clearly makes no sense for $x < 0$ unless p is an even integer; we exclude this case from consideration because it is the mirror image of the $x > 0$ case.

The trick to generate the x's is to make the change of variable defined by

$$dy = \lambda x^p dx \tag{6}$$

where λ is a positive constant to be determined. Eq. (5) then becomes

$$\frac{dN}{dy} = k/\lambda \tag{7}$$

which is similar to Eq. (1) except that y replaces x. Therefore we set $y = \alpha\hat{u} + \beta$, except that now the introduction of the constant λ affords the freedom to choose $\alpha = 1$ and $\beta = 0$, so that $y = \hat{u}$. Integrating Eq. (6) yields

$$y = ax^{p+1} - b \tag{8}$$

where $a \equiv \lambda/(p+1)$ and b is an integration constant to be determined. The algorithm is, therefore,

$$x = \left(\frac{\hat{u} + b}{a}\right)^{1/(p+1)} \tag{9}$$

The constants a (or, equivalently, λ) and b are determined by the requirements that $y = 0$ for $x = x_1$ and $y = 1$ for $x = x_2$, which yield

$$\frac{b}{a} = x_1^{p+1}, \qquad \frac{1}{a} = x_2^{p+1} - x_1^{p+1} \tag{10}$$

from which we get the explicit algorithm

$$x = \left[(x_2^{p+1} - x_1^{p+1})\hat{u} + x_1^{p+1}\right]^{1/(p+1)} \tag{11}$$

Note again that the normalization constant

$$K = k\int_{x_1}^{x_2} dx\, x^p = \frac{k(x_2^{p+1} - x_1^{p+1})}{p+1} \tag{12}$$

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plays no direct role (only K/k matters), and it must be finite.

A. Special cases (1): $x_1 = 0$ or $x_2 = \infty$.

These cases are possible as long as p is such that $K < \infty$, ie., as long as the distribution is normalizable. This condition implies the following restrictions on p:

$$x_1 = 0 \text{ and } x_2 < \infty : \; p > -1 \tag{13a}$$

$$x_1 > 0 \text{ and } x_2 = \infty : \; p < -1 \tag{13b}$$

$$x_1 = 0 \text{ and } x_2 = \infty : \; \text{impossible} \tag{13c}$$

B. Special case (2): $p = -1$.

One can take the limit $p \to -1$ by using l'Hôpital's rule applied to either (9) or (11) or, more simply, by starting directly from Eqs. (5) and (6). In this case $dy/dx = \lambda/x$ hence

$$y = a \ln x - b \tag{14}$$

where $a \equiv \lambda$. Imposing the conditions $y = 0$ when $x = x_1$ and $y = 1$ when $x = x_2$ yields

$$\frac{1}{a} = \ln\left(\frac{x_2}{x_1}\right), \qquad \frac{b}{a} = \ln x_1 \tag{15}$$

therefore, setting $y = \hat{u}$, we obtain

$$x = \exp\left(\frac{\hat{u} + b}{a}\right) = x_1 \left(\frac{x_2}{x_1}\right)^{\hat{u}} \tag{16}$$

Note that the value $p = -1$ forbids both choices $x_1 = 0$ and $x_2 = \infty$, because either one of these would make the distribution non-normalizable ($K = \infty$).

III. THE GENERAL CASE.

The problem is stated as follows: generate a one-dimensional random number distribution with a given probability density $\rho(x)$,

$$\frac{dN}{dx} = \rho(x) \tag{17}$$

in the interval $x_1 \le x \le x_2$. We require that $\rho(x)$ not vanish in any finite sub-interval within $[x_1, x_2]$. If it does, the problem reduces to generating random numbers from two (or more) additive disjoint distributions, which has a well-known solution.

Even though we are interested in a given x-interval, in many cases $\rho(x)$ will be defined in a larger region of the x axis. If $\rho(x)$ is given in analytic form, it is typically defined either in $-\infty < x < \infty$ or in $0 \le x < \infty$. Examples of distributions of the former kind are the Gaussian, $\rho(x) = k \exp(-(x - x_0)^2/2\sigma^2)$, the Lorentzian, $\rho(x) = k/((x - x_0)^2 + \gamma^2)$, etc. For these distributions x_1 may extend all the way to $-\infty$. Examples of the latter kind are the power density, $\rho(x) = kx^p$, the exponential, $\rho(x) = k \exp(-x)$, etc. For either kind, x_2 may extend to $+\infty$. The formulas below apply equally well to any such distributions. If $\rho(x)$ is only known in $x_1 \le x \le x_2$, see below.

The trick to generate the x's according to (17) is to make the change of variables

$$dy = \lambda \rho(x) dx \tag{18}$$

so that dN/dy is a uniform distribution in y, whose solution is fully described in Sec. I, namely $y = \alpha\hat{u} + \beta$ except that here, as in the power density case in Sec. II, we are free to choose $\alpha = 1$ and $\beta = 0$. To translate y into x, we integrate (18),

$$y = \lambda P(x) - b \tag{19}$$

where $P(x)$ is defined to be the integral of $\rho(x)$ relative to $x = 0$,

$$P(x) \equiv \int_0^x dx' \rho(x') \tag{20}$$

Eq. (19) yields

$$x = P^{-1}\left(\frac{y + b}{\lambda}\right) \tag{21}$$

where $P^{-1}(z)$ is the *functional inverse* (not to be confused with the algebraic inverse) of $P(x)$ (that is to say, if $z = P(x)$, then[1] $x = P^{-1}(z)$).

The constants λ and b follow from the requirements that $x = x_1$ when $y = 0$ and $x = x_2$ when $y = 1$, ie.

$$x_1 = P^{-1}\left(\frac{b}{\lambda}\right), \qquad x_2 = P^{-1}\left(\frac{1 + b}{\lambda}\right) \tag{22}$$

or, equivalently,

$$P_1 = \frac{b}{\lambda}, \qquad P_2 = \frac{1 + b}{\lambda} \tag{23}$$

where $P_i \equiv P(x_i)$, $i = 1, 2$. From here we easily obtain b and b/λ, hence Eq. (21) yields the general algorithm

$$x = P^{-1}\left((P_2 - P_1)\hat{u} + P_1\right) \tag{24}$$

Again, the normalization constant $K \equiv \int_{x_1}^{x_2} dx\, \rho(x) = P_2 - P_1$ must be finite.

If the density $\rho(x)$ is only defined, or only known, in the interval $x_1 \le x \le x_2$, or is not given in analytic form

[1] Note that there is a $1 \leftrightarrow 1$ correspondence between x and z owing to the monotonically increasing nature of $P(x)$.

(for example, it may be specified as a numerical table in this interval), then one may deal with an extended density defined by

$$\rho_{\rm ext}(x) = \begin{cases} \rho(x) & \text{if } x_1 \le x \le x_2 \\ 0 & \text{elsewhere} \end{cases} \tag{25}$$

and proceed as in the case above (a slight confusion may arise here because $P^{-1}(x)$ is not well defined outside $[x_1, x_2]$, although in practice this is not a problem). Equivalently, it is conceptually simpler to define $P(x)$ relative to $x = x_1$ rather than to $x = 0$, namely

$$P(x) \equiv \int_{x_1}^{x} dx' \rho(x') \,, \quad x_1 \le x \le x_2 \tag{26}$$

With this definition, $P_1 = 0$ and $P_2 = K$, hence the algorithm is now written[2]

$$x = P^{-1}(K\hat{u}) \tag{27}$$

Depending on the complexity of the expressions for P_1 and P_2, one may be able to simplify somewhat expressions (24) or (27) by the replacement $\hat{u} \to 1 - \hat{u}$ because $1-\hat{u}$ is as good a uniform random number in $[0, 1]$ as $\hat{u}$ is. With this replacement, however, $\hat{u} = 0(1)$ gets mapped onto $x = x_2(x_1)$ rather than $x_1(x_2)$, an immaterial difference in the algorithm.

Example: power density.

To show how the general formalism above applies to the power density we use Eq. (20) to obtain

$$z \equiv P(x) = k \int_0^x dx' x'^p = \frac{kx^{p+1}}{p+1} \tag{28}$$

therefore

$$x = P^{-1}(z) = \left[\frac{(p+1)z}{k}\right]^{1/(p+1)} \tag{29}$$

We now use

$$P_i = \int_0^{x_i} dx \, \rho(x) = \frac{kx_i^{p+1}}{p+1} \,, \quad i = 1, 2 \tag{30}$$

and, according to (24), insert $z = (P_2 - P_1)\hat{u} + P_1$ into (29), thus recovering the original expression (11).

Had we applied Eq. (26) instead of (20) we would have obtained

$$z \equiv P(x) = k \int_{x_1}^{x} dx' x'^p = \frac{k}{p+1}(x^{p+1} - x_1^{p+1}) \tag{31}$$

hence

$$x = P^{-1}(z) = \left[\frac{(p+1)z}{k} + x_1^{p+1}\right]^{1/(p+1)} \tag{32}$$

Following Eq. (27), we use $P_1 = 0$, $P_2 = K$ and $z = K\hat{u}$ in (32), yielding exactly the same result as above, namely Eq. (11).

IV. REMARK.

This algorithm presented here is most efficient when (a) $P(x)$ is obtainable in analytic form, and (b) its functional inverse has a simple analytic form. If these conditions are not satisfied, the next best option is to tabulate both $P(x)$ and $P^{-1}(x)$ in the interval $x_1 \le x \le x_2$ and use table interpolation; this option works well if $P(x)$ is smooth enough and the interval is finite. If none of the above conditions are met, the conventional alternative to generating random numbers is the Monte Carlo ("accept-reject") method. This method has the advantages of great simplicity and straightforward extension to higher dimensions, but it may have efficiency problems.

V. OTHER EXAMPLES.

Here we provide a list of examples that might be useful, without the detailed derivation. For each case we provide $\rho(x)$, the relevant interval, and any applicable condition on the parameters that might appear in $\rho(x)$.

1. $\rho(x) = k \exp(-(x - x_0)^2/2\sigma^2)$, $x_1 \le x \le x_2$.
   Conditions: x_0 arbitrary, $\sigma > 0$.

$$x = x_0 + \sqrt{2}\sigma \, \mathrm{erf}^{-1}((P_2 - P_1)\hat{u} + P_1) \tag{33a}$$

$$P_i \equiv \mathrm{erf}\left(\frac{x_i - x_0}{\sqrt{2}\sigma}\right) \,, \quad i = 1, 2 \tag{33b}$$

   Note that if $x_1 = -\infty$ and $x_2 = +\infty$ you get

$$x = x_0 + \sqrt{2}\sigma \, \mathrm{erf}^{-1}(2\hat{u} - 1) \tag{34}$$

2. $\rho(x) = k/(x^2 + \gamma^2)$, $-\infty < x < \infty$.
   Conditions: $\gamma > 0$.

$$x = \gamma \tan\left[\frac{(2\hat{u} - 1)\pi}{2}\right] \tag{35}$$

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[2] This variant of defining $P(x)$ relative to $x = x_1$ rather than to $x = 0$ is always valid, whether $\rho(x)$ is given in analytic form or not. Even though we use the same notation for $P(x)$ in (20) and (26), it should be clear that these are different functions, as are their functional inverses (24) and (27), respectively. Also, expression (26) for $P(x)$ contains an implicit dependence on x_1 which we suppress for notational compactness.

3. $\rho(x) = k \sin \pi x$, $0 \le x \le 1$

$$x = \frac{1}{\pi} \cos^{-1}(2\hat{u} - 1) \tag{36}$$

where the function $\cos^{-1} x$ is defined in $0 \le x \le \pi$.

4. $\rho(x) = k \cos(\pi x/2)$, $-1 \le x \le 1$.

$$x = \frac{2}{\pi} \sin^{-1}(2\hat{u} - 1) \tag{37}$$

where the function $\sin^{-1} x$ is defined in $-\pi/2 \le x \le \pi/2$.

5. $\rho(x) = k(1 - x^2)$, $-1 \le x \le 1$.

$$x = 2 \sin\left(\tfrac{1}{3} \sin^{-1}(2\hat{u} - 1)\right) \tag{38}$$

where the function $\sin^{-1} x$ is defined in $-\pi/2 \le x \le \pi/2$.

6. $\rho(x) = kx^{p-1} \exp(-x^p)$, $0 \le x < \infty$.

Conditions: $p \ne 0$.

$$x = (-\ln \hat{u})^{1/p} \tag{39}$$

7. $\rho(x) = kx^{p-1} e^{-x}$, $0 \le x < \infty$.

Conditions: $p > 0$.

$$x = P^{-1}(p, \hat{u}) \tag{40}$$

where $P^{-1}(p, x)$ is the functional inverse (in x) of the incomplete gamma function $P(p, x)$ defined by

$$P(p, x) = \frac{1}{\Gamma(p)} \int_0^x dt\, t^{p-1} e^{-t}\,, \quad p > 0,\ x \ge 0 \tag{41}$$

If the desired interval in x is $x_1 \le x \le x_2$ where $0 \le x_1 < x_2 < \infty$, then

$$x = P^{-1}(p, (P_2 - P_1)\hat{u} + P_1) \tag{42a}$$

$$P_i \equiv P(x_i, p)\,, \quad i = 1, 2 \tag{42b}$$

8. $\rho(x) = ke^{-cx}$, $0 \le x < \infty$.

Conditions: $c > 0$. This is a special case of either of the above two examples, obtained by setting $p = 1$.

$$x = -\frac{1}{c} \ln \hat{u} \tag{43}$$

If the desired interval in x is $x_1 \le x \le x_2$, then

$$x = -\frac{1}{c} \ln\left[E_1 - (E_1 - E_2)\hat{u}\right] \tag{44a}$$

$$E_i \equiv e^{-cx_i}\,, \quad i = 1, 2 \tag{44b}$$

9. $\rho(x) = kx^{\mu-1}(1 - x)^{\nu-1}$, $0 \le x \le 1$.

Conditions: $\mu, \nu > 0$.

$$x = \beta^{-1}(\hat{u}, \mu, \nu) \tag{45}$$

where $\beta^{-1}(x, \mu, \nu)$ is the functional inverse (in x) of the normalized incomplete beta function $\beta(x, \mu, \nu)$, defined by

$$\beta(x, \mu, \nu) = \frac{\Gamma(\mu + \nu)}{\Gamma(\mu)\Gamma(\nu)} \int_0^x dt\, t^{\mu-1}(1 - t)^{\nu-1}\,, \qquad 0 \le x \le 1,\ \mu > 0,\ \nu > 0 \tag{46}$$

If the desired interval in x is $x_1 \le x \le x_2$ where $0 \le x_1 \le x_2 \le 1$, then

$$x = \beta^{-1}((\beta_1 - \beta_1)\hat{u} + \beta_1, \mu, \nu) \tag{47a}$$

$$\beta_i \equiv \beta(x_i, \mu, \nu)\,, \quad i = 1, 2 \tag{47b}$$

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