# Compact Complex Expressions for the Electric Field of 2-D Elliptical Charge Distributions*†

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(Dated: Revised version: June 1st, 1994; minor corrections: March 21, 1997; additions: May 21, 2007)

We present a formula for the analytic calculation of the electric field in complex form for two-dimensional charge distributions with elliptical contours, in the absence of boundary conditions except at infinity. The formula yields compact and practical expressions for a significant class of distributions. The fact that the electric field vanishes inside an elliptical shell follows as a straightforward consequence of Cauchy's theorem. The known expressions for the field inside and outside a uniformly-charged ellipse are recovered in simple, concise form. Similarly, the expression for the field of a Gaussian distribution is found in a straightforward way as a special case of the more general formula. We also present a brief discussion of more complicated distributions.

## I. INTRODUCTION.

It has long been recognized that a large class of complicated problems in two-dimensional electrostatics (and magnetostatics) in free space can be solved in a compact and elegant fashion by replacing the ordinary plane with coordinates $(x, y)$ by the complex plane $x + iy$. The reason for the great advantage of the complex plane can be succinctly stated as follows: if we confine our attention to a charge-free region of space, then the relevant Maxwell's equations to be solved are

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \times \mathbf{E} = 0$$

subject to certain boundary conditions. In terms of the field components, these equations read

$$\frac{\partial E_x}{\partial x} = -\frac{\partial E_y}{\partial y}, \qquad \frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}$$

which are nothing but the Cauchy-Riemann conditions for the complex conjugate $\bar{E}$ of the "complex electric field," defined by

$$E \equiv E_x + iE_y\,. \tag{1}$$

A fundamental theorem of complex analysis then implies that $\bar{E}$ is an analytic function of the complex variable $z \equiv x + iy$ or, equivalently, that the field itself $E$ is an analytic function of $\bar{z}$,

$$E = E(\bar{z}). \tag{2}$$

There are two consequences from this analyticity property: (1), the complex electric field depends on $x$ and $y$ only through the combination $\bar{z} \equiv x - iy$ (i.e., the combination $x + iy$ is not allowed); and (2), the method of conformal mappings may then be used to transform a complicated boundary condition geometry into a new, simpler geometry in which it is easy to find an analytic function that satisfies the condition. Since conformal mappings preserve analyticity, the solution for the electric field satisfying the original boundary conditions is obtained by simply applying the inverse of the mapping.

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* Published: Am. J. Phys. **62**(12), 1994, pp. 1134–1140.
†Work supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, High Energy Division, of the U.S. Department of Energy under Contract no. DE-AC03-76SF00098.
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Now if the problem to be solved involves free charges, the divergence equation has a nonzero right-hand side,

$$\nabla \cdot \mathbf{E} = 4\pi\rho(\mathbf{x}) \tag{3}$$

and therefore the complex electric field does not satisfy the Cauchy-Riemann conditions hence it is not an analytic function in the region where $\rho \neq 0$. As a result, the method of conformal mappings is not applicable, although the complex representation of the field can still be used profitably in many cases due to the compactness of the notation [1].

For charge densities with elliptical contours, the conventional approach is based upon solving Poisson's equation for the potential in ellipsoidal coordinates [2]. For this class of densities we show here that Cauchy's theorem allows an alternative calculation of the field in a much simpler form and yields compact expressions that are convenient to use in algebraic or numerical computations. Although this class is obviously a limited one, it is found in many applications in beam physics [3–7] and in astronomy [8]. Our method yields a single formula valid in free space and also in the regions where charges are present. The formalism applies naturally to the electric field itself rather than to the electrostatic potential.[1] It seems possible to generalize the method to charge distributions whose contours are more complicated provided these contours are nonintersecting closed curves, although it is clear that the basic case corresponds to ellipses. The disadvantage of the method is that it is restricted to two-dimensional distributions.

In Sec. 2 we present the general setup of the calculation of the complex electric field in the presence of a charge density with elliptical contours. In Sec. 3 we carry out the most basic case, namely that in which the density is a delta function over an elliptical shell. The well-known result that the field vanishes inside the shell follows as a simple consequence of Cauchy's theorem. The complex electric field for a general charge distribution with elliptical contours is the superposition of the field for the previous case weighted by the charge density. In Sec. 4 we carry out several examples of this superposition for specific charge densities. In particular, we recover in very simple way the known results for the cases of a uniformly-charged ellipse and for a Gaussian distribution. We also present a general discussion of polynomial distributions. In Sec. 5 we present some remarks, and in Sec. 6 our conclusions.

## II. GENERAL SETUP OF THE CALCULATION.

We consider a static three-dimensional charge distribution with translation invariance along the direction perpendicular to the $(x, y)$-plane, and define $\lambda$ to be the charge per unit length along the translation-invariant axis. The solution of Eq. (3) for the case of a single line of charge located at the origin, which we label with a subscript 0, is

$$\mathbf{E}_0(\mathbf{x}) = 2\lambda \frac{\mathbf{x}}{|\mathbf{x}|^2}$$

where $\mathbf{x}$ is a two-dimensional coordinate vector with components $(x, y)$. The general solution for an arbitrary translation-invariant charge density $\rho(\mathbf{x})$ is obtained from the superposition principle,

$$\mathbf{E}(\mathbf{x}) = 2 \int d^2\mathbf{x}' \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^2} \rho(\mathbf{x}') \,. \tag{4}$$

From the definition (1), these equations read, in complex form,

$$E_0(\mathbf{x}) = \frac{2\lambda}{\bar{z}}$$

and

$$E(\mathbf{x}) = 2 \int d^2\mathbf{x}' \frac{\rho(\mathbf{x}')}{\bar{z} - \bar{z}'} \tag{5}$$

respectively, where the bar denotes complex conjugation, and $z' \equiv x' + iy'$. It should be noted that $E_0$ exhibits the analyticity property (2) for all $z \neq 0$, as it should. Clearly, Eq. (5) is completely equivalent to Eq. (4). The trick to simplify (5) is to reduce the two-dimensional integral to a one-dimensional integral over a simple (i.e., nonintersecting)

[1] Our formalism is much more complicated if applied to the potential due to its logarithmic nature. In three dimensions it is obviously simpler to work with the potential than with the field.

closed curve and then to take advantage of Cauchy's theorem. Obviously the success of this method depends on the properties of the charge density $\rho$.

In this note we are only concerned with a specific class of density functions, namely those with elliptical contours. That is to say, we assume that, with an appropriate choice of origin and orientation of the coordinate axes, $\rho$ depends on $x$ and $y$ only through the combination

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \tag{6}$$

rather than on $x$ and $y$ separately. We assume, without any loss of generality, that $a \geq b$; furthermore, if $\rho$ is a rigorously localized distribution, we define $a$ and $b$ to be the semi-axes of the largest ellipse that encloses charge, namely that $\rho = 0$ for $x^2/a^2 + y^2/b^2 > 1$. If $\rho$ extends over all space (such as in the case of the Gaussian distribution), the parameters $a$ and $b$ can be best interpreted as (or traded off for) the rms sizes of the distribution $\sigma_x$ and $\sigma_y$, respectively, by using Eq. (8) below.

Eq. (6) implies that the charge density of an elliptical distribution is generally written

$$\rho\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = \int\limits_0^\infty dt\,\rho(t)\,\delta\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - t\right). \tag{7}$$

With the change of variables described in Sec. 3 it is straightforward to show that the total linear charge density $\lambda$ of the distribution, which we assume finite, is

$$\lambda \equiv \int d^2\mathbf{x}\rho(\mathbf{x}) = \pi ab \int\limits_0^\infty dt\,\rho(t)$$

and that the rms sizes are given by

$$\frac{\sigma_x^2}{a^2} = \frac{\sigma_y^2}{b^2} = \frac{1}{2}\int\limits_0^\infty dt\,t\hat{\rho}(t) \tag{8}$$

where we have introduced the dimensionless density $\hat{\rho}(t) \equiv (\pi ab/\lambda)\rho(t)$ which is normalized to unity,

$$\int\limits_0^\infty dt\,\hat{\rho}(t) = 1. \tag{9}$$

Inserting Eq. (7) into Eq. (5) and interchanging the order of the integration,[2] the complex electric field for a general elliptical distribution becomes

$$E(\mathbf{x}) = \int\limits_0^\infty dt\,\hat{\rho}(t) \int d^2\mathbf{x}'\,\frac{2}{\bar{z} - \bar{z}'}\,\frac{\lambda}{\pi ab}\,\delta\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - t\right). \tag{10}$$

This analysis for density functions with elliptical contours [5] is applicable to the electrostatic potential or to the electric field in two or three dimensions. Obviously, however, the complex-field formalism is peculiar to two dimensions. The subexpression $\int d^2\mathbf{x}'(\cdots)$ in Eq. (10) is nothing but the electric field produced by a delta-function density over an elliptical shell of "radius" $t$, while the integral $\int dt\,\hat{\rho}(t)(\cdots)$ is the superposition of the fields produced by all shells of different radii with a weight given by $\hat{\rho}(t)$. In Sec. 3 we will evaluate the field produced by a single delta-function shell, and in Sec. 4 we will carry out the superposition integral for various densities $\hat{\rho}(t)$.

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[2] This step requires $\rho$ to be integrable, namely $|\lambda| < \infty$. If $\rho$ is not positive-definite or negative-definite, the requirement is that $\rho$ must be absolutely integrable, namely $\int d^2\mathbf{x}\,|\rho(\mathbf{x})| < \infty$.

### III. THE BASIC CASE: DELTA-FUNCTION ELLIPTICAL SHELL.

We present here the calculation of the complex electric field for the basic case, namely that of an infinitesimally thin elliptical shell. For simplicity we temporarily set $t = 1$ in the expression for the charge density, which then reads

$$\rho(\mathbf{x}) = \frac{\lambda}{\pi ab}\,\delta\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \tag{11}$$

and take the complex conjugate of the electric field, Eq. (10), so that

$$\bar{E}(\mathbf{x}) = \frac{2\lambda}{\pi ab}\int d^2\mathbf{x}'\,\frac{\delta(x'^2/a^2 + y'^2/b^2 - 1)}{z - z'}$$

(we will recover the result for $t \neq 1$ later in this section). The change of variables $x' = ar'\cos\phi'$, $y' = br'\sin\phi'$ implies

$$\int d^2\mathbf{x}'(\cdots) = ab\int_0^{2\pi} d\phi' \int_0^{\infty} r'dr'(\cdots)\,.$$

The integral over $r'$ is straightforward, yielding

$$\bar{E}(\mathbf{x}) = \frac{\lambda}{\pi}\int_0^{2\pi}\frac{d\phi'}{z - z'(\phi')} \tag{12}$$

where $z'(\phi') \equiv a\cos\phi' + ib\sin\phi'$ represents the elliptical charge shell in complex parametric form. A further change of variable, defined by $\zeta = \exp(i\phi')$, transforms Eq. (12) into a Cauchy-type integral over the unit circle in the complex-$\zeta$ plane,

$$\bar{E}(\mathbf{x}) = \frac{2\lambda}{i\pi}\oint_{|\zeta|=1}\frac{d\zeta}{2\zeta z - (a+b)\zeta^2 - a + b} \tag{13}$$

which can be done by the method of residues.

The poles $\zeta_\pm$ of the integrand are easily found to be

$$\zeta_\pm(z) = \frac{1}{a+b}\left[z \pm \sqrt{z^2 - a^2 + b^2}\,\right]. \tag{14}$$

Now in order to provide a well-defined result, we need to make the square-root function in this expression unambiguous by means of an appropriate Riemann cut in the $z$-plane. There are two square-root type cuts that emanate out of the foci of the ellipse, which are located on the real axis at $x = \pm(a^2 - b^2)^{1/2}$ . The cuts are, in principle, almost arbitrary: any pair of nonintersecting semi-infinite curves emanating out of the foci of the ellipse renders the square root in (14) unambiguous. It turns out that the requirement that the electric field be an unambiguous, odd-parity function of $z$ implies that there is a unique specification for the topology of the branch cuts. For the time being, however, we proceed with the evaluation of Cauchy's integral, keeping in mind that such a specification will be made explicit below.

We first note that the poles satisfy the relation

$$\zeta_+\zeta_- = \frac{a-b}{a+b}$$

which is a positive real number $< 1$. This implies that at least one of these poles ($\zeta_-$) is always contained inside the unit circle, regardless of the value of $z$. The other pole, ($\zeta_+$), may or may not be inside the unit circle, depending on the value of $z$. More specifically, one can show that the functions $\zeta_\pm(z)$ are conformal mappings with the following properties: The mapping $\zeta_+(z)$ maps the ellipse $z = a\cos\phi + ib\sin\phi$ to the unit circle $|\zeta| = 1$, the exterior of the ellipse to the exterior of the unit circle and the interior of the ellipse to the annulus $[(a-b)/(a+b)]^{1/2} < |\zeta| < 1$. The mapping $\zeta_-$ maps the ellipse $z = a\cos\phi + ib\sin\phi$ to the circle $|\zeta| = (a-b)/(a+b)$, the exterior of the ellipse to the disk $0 < |\zeta| < (a-b)/(a+b)$, and the interior of the ellipse to the annulus $(a-b)/(a+b) < |\zeta| < [(a-b)/(a+b)]^{1/2}$.

For the calculation of the field we first rewrite Eq. (13) in the form

$$\bar{E}(\mathbf{x}) = \frac{2i\lambda}{\pi(a+b)} \oint_{|\zeta|=1} \frac{d\zeta}{(\zeta-\zeta_+)(\zeta-\zeta_-)}$$

and consider two cases separately:

(1) If the observation point $\mathbf{x}$ is inside the elliptical charge shell, i.e., if $z$ is inside the ellipse, then both poles $\zeta_+$ and $\zeta_-$ are inside the unit circle $|\zeta| = 1$, and Cauchy's theorem yields

$$\bar{E}(\mathbf{x}) = 0 \qquad \text{(inside)}.$$

(2) If the observation point $\mathbf{x}$ is outside the elliptical charge shell, then only $\zeta_-$ is inside the unit circle, and Cauchy's theorem yields

$$\bar{E}(\mathbf{x}) = \frac{2i\lambda}{\pi(a+b)} \cdot \frac{2\pi i}{\zeta_- - \zeta_+} = \frac{2\lambda}{\sqrt{z^2 - a^2 + b^2}} \qquad \text{(outside)}. \tag{15}$$

This completes the calculation except that, as mentioned above, we need to specify the Riemann cuts in the $z$-plane in order to make $(z^2 - a^2 + b^2)^{1/2}$ well-defined. We first note that the original expression, Eq. (13), defines the electric field to be an unambiguous function that is odd under the parity transformation and also odd under reflections. The odd-parity property, $E(-x,-y) = -E(x,y)$, can be proven by making the replacement $z \to -z$ in Eq. (13) followed by the change of integration variable $\zeta \to -\zeta$ and using the identity $\oint d(-\zeta) f(-\zeta) = -\oint d\zeta f(\zeta)$, valid for any function $f(\zeta)$. The odd-reflection property, which is compactly stated $\bar{E}(x,y) = E(x,-y)$, follows from taking the complex conjugate of Eq. (13) and using the identity $\oint d\bar{\zeta} f(\bar{\zeta}) = -\oint d\zeta f(\zeta)$. These two properties of the field are true for any charge density that is even under parity and under reflections, such as (11).

The expression (15), on the other hand, has the appearance of being of even parity, since it depends on $z$ only through $z^2$. This is misleading: in general, determining the parity of a function that involves branch cuts is a topological problem because it depends on *which path* one chooses to go from $z$ to $-z$. Clearly, there are only two topologically distinct possibilities to define the Riemann cut structure for this square root: (a) the two cuts merge, joining together the two foci of the ellipse, as shown in Fig. 1a, and (b) the cuts emanate out of the foci of the ellipse and extend all the way to infinity, as in Fig. 1b. One can easily prove (see Fig. 2) that the cut structure (a) makes $(z^2 - a^2 + b^2)^{1/2}$ an odd-parity function of $z$, while structure (b) makes it an even-parity function. Therefore the cut structure (a) provides the correct specification for the electric field.

Another way to establish that cut (a) is the correct answer is to note that the structure (b) would lead to unphysical discontinuities of the electric field across the cut in the region outside the ellipse, thus violating the fact that Eq. (13) defines the field in a smooth and unambiguous fashion. In the region inside the ellipse the field vanishes identically, hence no discontinuity arises from a cut joining the two foci.

The cut is almost arbitrary: any nonintersecting line joining the foci that is wholly contained within the ellipse will do. However, in practical applications, it may be convenient to assume the cut to be a straight line, as shown in Fig. 1a.

The odd-reflection property implies that the real part of the square root $(z^2-a^2+b^2)^{1/2}$ is odd under $(x,y) \to (-x,y)$ and even under $(x,y) \to (x,-y)$, while the imaginary part has the opposite properties. With a cut of type (a), and defining $u \equiv x^2 - y^2 - a^2 + b^2$, the parity and reflection properties are made explicit by the following formula, valid for the region outside the ellipse:

$$\sqrt{z^2 - a^2 + b^2} = \frac{\text{sign}(x)}{\sqrt{2}} \sqrt{u + \sqrt{u^2 + (2xy)^2}} + i\frac{\text{sign}(y)}{\sqrt{2}} \sqrt{-u + \sqrt{u^2 + (2xy)^2}}. \tag{16}$$

Finally, we generalize to the case in which $t \neq 1$ corresponding to

$$\rho(\mathbf{x}) = \frac{\lambda}{\pi ab}\, \delta\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - t\right).$$

The result for the field is obtained from the expression for $t = 1$, Eq. (15), by the replacements $(a,b) \to \sqrt{t}(a,b)$. Using well-known properties of the delta function, one finds

$$E(\mathbf{x}) = 2\lambda\, \frac{\theta(x^2/a^2 + y^2/b^2 - t)}{\sqrt{\bar{z}^2 - t(a^2 - b^2)}} \tag{17}$$

where we have taken the complex conjugate and have used the step function $\theta(\cdots)$ to make explicit the fact that the field vanishes inside the elliptical shell. The square root in this last expression is made well-defined by a branch cut of the type shown in Fig. 1a except that, for $t \neq 1$, the branch points are located at $\pm[t(a^2 - b^2)]^{1/2}$.

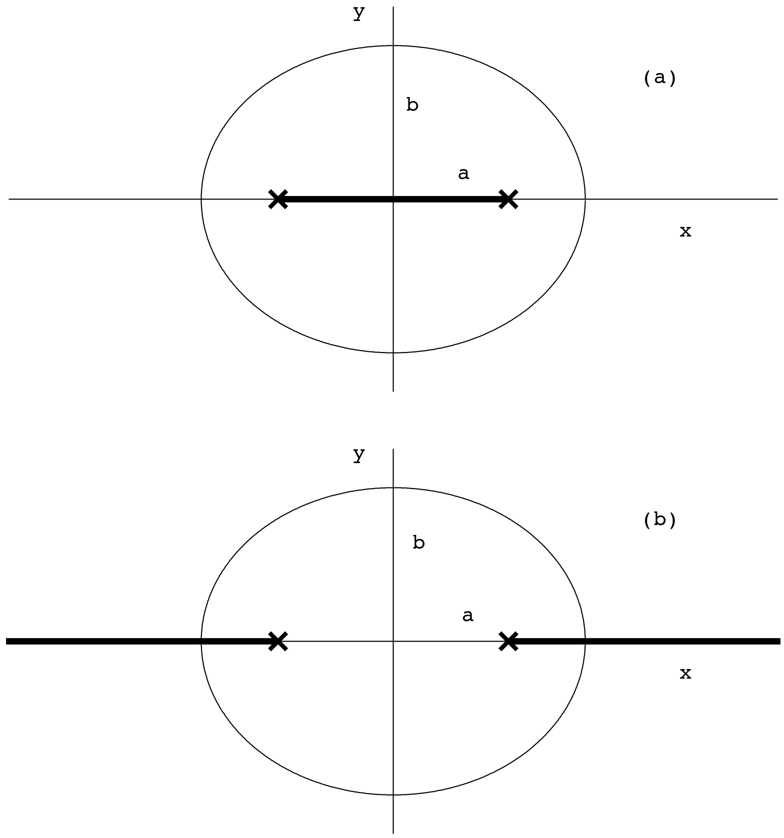


FIG. 1: (a): The square root $f(z) \equiv (z^2 - g^2)^{1/2}$, where $g \equiv (a^2 - b^2)^{1/2}$, is rendered an unambiguous odd-parity function of $z$ for all $z$ by joining into a single straight line the branch cuts emanating out of the foci of the ellipse, indicated by crosses at $x = \pm g$. (b): If the cuts extend out to infinity along the real axis, $f(z)$ is an even-parity function of $z$ for all $z$, with unphysical discontinuities across the real axis. Case (a) gives the correct answer for the complex electric field. The cuts need not be straight lines, although this is the simplest choice. See Fig. 2 for a detailed explanation of the even/odd parity of $f(z)$ implied by the two types of branch cuts.

## IV. APPLICATIONS.

### A. General remarks and limiting forms.

In general, as discussed in the introduction, the complex electric field is not a function of $\bar{z}$ alone because it is not, in general, an analytic function. It turns out, however, that in all but the simplest cases, the electric field can be expressed very compactly in terms of the auxiliary complex variables

$$\xi \equiv \frac{x}{a} + i\frac{y}{b} \qquad \text{and} \qquad \omega \equiv \frac{bx}{a} + i\frac{ay}{b}$$

and their complex conjugates, in addition to $z \equiv x + iy$. Obviously these variables are not independent; their relationship is most conveniently expressed by the easily-proven identities

$$z - \omega = (a - b)\bar{\xi} \tag{18a}$$
$$z + \omega = (a + b)\xi \tag{18b}$$

and their complex-conjugate counterparts. A relation that is particularly useful follows from multiplying these two together,

$$(a^2 - b^2)|\xi|^2 = \bar{z}^2 - \bar{\omega}^2 = z^2 - \omega^2. \tag{19}$$

As an example of the simplification achieved with the use of these auxiliary variables, we note that the field just

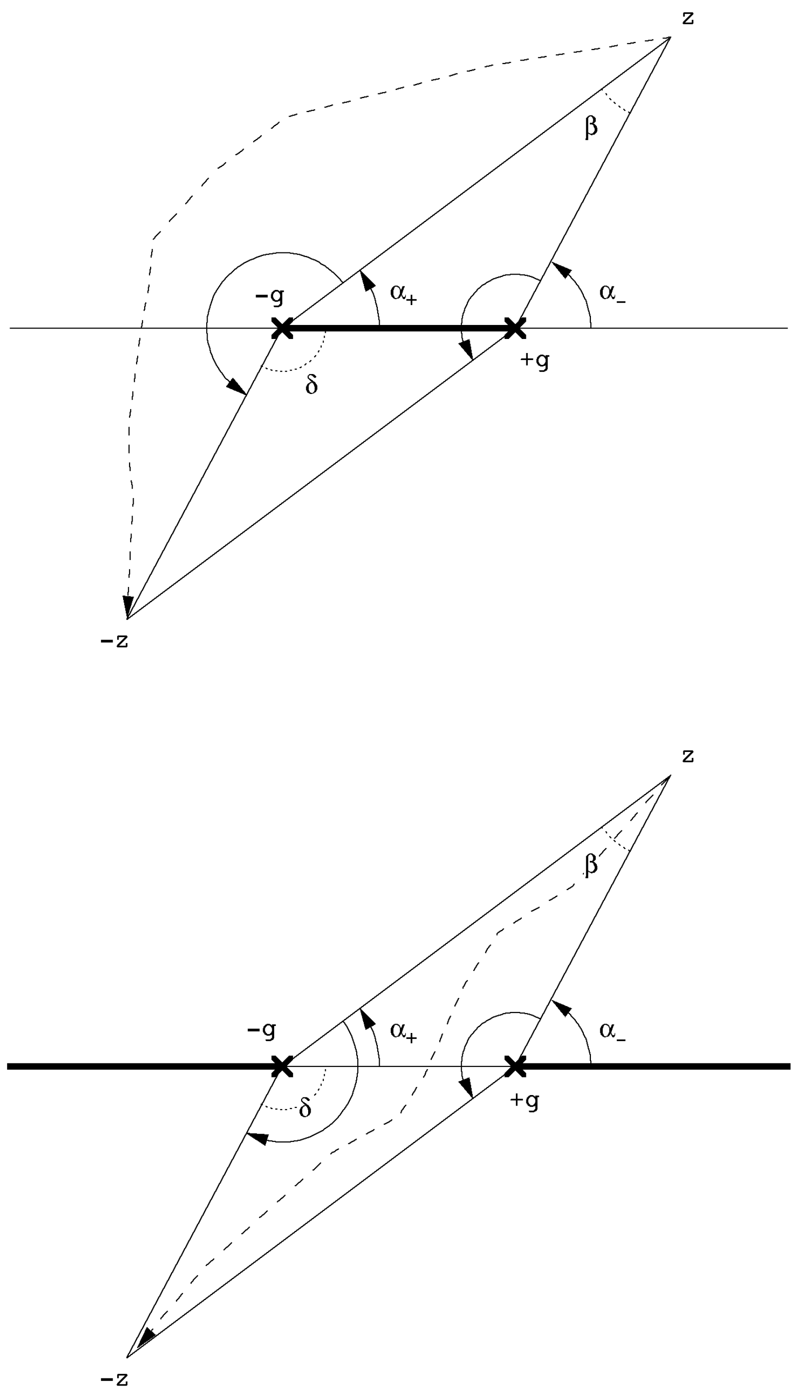


FIG. 2: The parity of the function $f(z) = (z^2 - g^2)^{1/2}$ under the transformation $z \to -z$ depends on the topology of the Riemann cut structure. Defining $g \equiv (a^2 - b^2)^{1/2}$ and $z \pm g = r_\pm \exp(i\alpha_\pm)$, where $r_\pm = |z \pm g|$, the function $f(z)$ is written $f(z) = (r_+ r_-)^{1/2} \exp(i(\alpha_+ + \alpha_-)/2)$. Top: for branch cut type (a), when going from $z$ to $-z$ along the dashed line, the angle $\alpha_+$ changes according to $\alpha_+ \to \alpha_+ + \delta + 2\beta + \alpha_+$, while $\alpha_- \to \alpha_- + \delta + \alpha_+$. Using $\delta = \pi - \alpha_-$ and $\beta = \alpha_- - \alpha_+$ we get $\alpha_+ + \alpha_- \to \alpha_+ + \alpha_- + 2\pi$, hence $f(z) \to -f(z)$ (the same result is obtained when the dotted-line path is to the right of $+g$). Bottom: for a branch cut type (b), the path from $z$ to $-z$ goes in between the two branch points at the foci. In this case $\alpha_-$ changes in the same way as for the previous case, but $\alpha_+ \to \alpha_+ - \alpha_+ - \delta$ hence $\alpha_+ + \alpha_- \to \alpha_+ + \alpha_-$, hence $f(z) \to +f(z)$. In the limit $g \to 0$, the branch cut type (a) disappears and we obtain $f(z) = z$ which is everywhere analytic. With the branch cut type (b) the cuts remain when $g = 0$, and we obtain $f(z) = \epsilon(y)z$ where $\epsilon(y)$ is the sign of $y$. This limiting function is analytic everywhere except on the real axis, where $f'(z)$ is discontinuous.

outside the elliptical shell is

$$E(\mathbf{x}) = \frac{2\lambda}{\bar{\omega}} \qquad \text{(just outside)}$$

which is obtained by inserting the identity (19) into Eq. (17) and noting that the charge shell is defined by $|\xi|^2 = t$.

By combining Eqs. (10) and (17) we arrive at the general expression for the complex electric field for an elliptical distribution,

$$E(\mathbf{x}) = 2\lambda \int_0^{|\xi|^2} \frac{dt\, \hat{\rho}(t)}{\sqrt{\bar{z}^2 - t(a^2 - b^2)}} \tag{20}$$

which constitutes the central result of this article. For each $t$, the cut is a nonintersecting line wholly contained within the ellipse $x^2/a^2 + y^2/b^2 = t$, joining the two foci at $\pm[t(a^2 - b^2)]^{1/2}$. Thus, overall, the cut extends out to $\pm|\xi|(a^2 - b^2)^{1/2}$. However, for a rigorously localized charged distribution, the cut extends only out to $\pm(a^2 - b^2)^{1/2}$ when the observation point is outside the region with charge, according to the discussion following Eq. (6). The integral can be carried out analytically for a significant class of weight functions $\hat{\rho}(t)$, of which we will provide a few examples below (the thin-shell case, Eq. (17), is recovered by setting $\hat{\rho}(t) = \delta(t - 1)$).

If $\hat{\rho}(t)$ is finite at $t = 0$, one can easily find from Eq. (20) the leading expression for the field at the origin,

$$E(\mathbf{x}) \to \frac{4\lambda\hat{\rho}(0)}{a + b}\,\xi \quad \text{as } |z| \to 0. \tag{21}$$

If $\hat{\rho}(t)$ has finite extent, or if it falls sufficiently rapidly at large distance, then Eq. (20) and the normalization property (9) imply that

$$E(\mathbf{x}) \to \frac{2\lambda}{\bar{z}} \quad \text{as } |z| \to \infty \tag{22}$$

so that the field approaches that of a line of charge, as it should be expected. Nonleading corrections to this limit, which constitute the multipole expansion of the field, can be easily obtained from Eq. (20) by Taylor-expanding the integrand at infinity.[3]

### B. Round charge distribution.

If the charge distribution is round rather than elliptical, Eq. (20) yields, for the field at distance $r$ from the center,

$$E(\mathbf{x}) = \frac{2\lambda}{\bar{z}} \int_0^{r^2/a^2} dt\, \hat{\rho}(t)$$

which is the result one obtains in a straightforward manner from Gauss' theorem (in arriving at this result, we have used $\sqrt{\bar{z}^2 - t(a^2 - b^2)} = \bar{z}$ in the limit when $a = b$, as explained in the caption of Fig. 2). In particular, if the charge distribution is a thin round shell, this yields the well-known result

$$E(\mathbf{x}) = \begin{cases} 0 & \text{inside} \\ E_0(\mathbf{x}) & \text{outside.} \end{cases}$$

### C. Uniformly charged ellipse.

In this case the charge density is

$$\rho(\mathbf{x}) = \begin{cases} \dfrac{\lambda}{\pi ab} & \text{if } x^2/a^2 + y^2/b^2 \le 1 \\ 0 & \text{if } x^2/a^2 + y^2/b^2 > 1 \end{cases}$$

[3] The resulting multipole expansion is a convergent series only if the charge distribution is of finite extent; otherwise, the expansion is asymptotic.

so that $\hat{\rho}(t) = \theta(1-t)$. Then the complex field is given by

$$E(\mathbf{x}) = 2\lambda \int_0^T \frac{dt}{\sqrt{\bar{z}^2 - t(a^2-b^2)}}$$

where the top limit of the integral is $T = \min(1, |\xi|^2)$. The integral is elementary: if the observation point is inside the ellipse, then $T = |\xi|^2$ and we obtain

$$\begin{aligned} E(\mathbf{x}) &= \frac{4\lambda}{a^2-b^2}\left[\bar{z} - \sqrt{\bar{z}^2 - |\xi|^2(a^2-b^2)}\right] \\ &= \frac{4\lambda}{a^2-b^2}\left[\bar{z} - \bar{\omega}\right] \\ &= \frac{4\lambda}{a+b}\,\xi \qquad \text{(inside).} \end{aligned} \tag{23}$$

If the observation point is outside the ellipse, then $T = 1$ and the result is

$$E(\mathbf{x}) = \frac{4\lambda}{a^2-b^2}\left[\bar{z} - \sqrt{\bar{z}^2 - a^2 + b^2}\right] \tag{24a}$$

$$= \frac{4\lambda}{\bar{z} + \sqrt{\bar{z}^2 - a^2 + b^2}} \qquad \text{(outside).} \tag{24b}$$

In arriving at the final result in Eq. (23) for the field inside the ellipse we have used the identity (19). Notice that this expression is manifestly devoid of possible discontinuities or ambiguities, as it should be according to the earlier discussion on the nature of the cut. Although the two expressions (24a) and (24b) for the field outside the ellipse are mathematically equivalent, (24b) has the advantage that it is manifestly regular in the round-beam limit, $a \to b$, and that it has a straightforward long-distance limit, $|z| \to \infty$. Therefore this second form might be preferable in numerical computations. Notice also that the field in the charge-free region, Eqs. (24), is an analytic function of $\bar{z}$, as it should be.

At the edge of the ellipse the field must be continuous, and therefore the two expressions (23) and (24) must coincide. This can be verified as follows: we first note that the edge of the ellipse is defined by $|\xi|^2 = 1$, so that the identity (19) reduces to $a^2 - b^2 = \bar{z}^2 - \bar{\omega}^2$. Substituting this into Eq. (24a) and using (18a) yields

$$E(\mathbf{x}) = \frac{4\lambda}{a^2-b^2}(\bar{z} - \bar{\omega}) = \frac{4\lambda}{a+b}\,\xi \qquad \text{(edge)}$$

which agrees with the expression for the field inside the ellipse, Eq. (23).

Inside the ellipse, Eq. (23) shows that the field is linear, with the $x$-component given by

$$E_x = \frac{4\lambda}{a(a+b)}\,x \qquad \text{(inside).}$$

This linearity property is well known [1–4] and is also true for a three-dimensional ellipsoid. Outside the ellipse, Eqs. (16) and (24a) yield, for the $x$-component,

$$E_x = \frac{4\lambda}{a^2-b^2}\left[x - \frac{\text{sign}(x)}{\sqrt{2}}\sqrt{u + \sqrt{u^2 + (2xy)^2}}\right], \tag{25}$$

while the $y$-component can be obtained from this by exchanging $x \leftrightarrow y$ and $a \leftrightarrow b$. The real counterpart of Eq. (24b) is, of course, more complicated. The compactness and relative simplicity of the complex form, Eq. (24a), are obvious when compared to this real form. If we specialize Eq. (24b) to the real axis, we obtain, for $x > a$,

$$E_x = \frac{4\lambda}{x + \sqrt{x^2 - a^2 + b^2}}$$

which agrees with the known result [4].

### D. Gaussian charge density.

In this case the charge density is

$$\rho(\mathbf{x}) = \frac{\lambda}{2\pi\sigma_x\sigma_y} \exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right)$$

so that $\hat{\rho}(t) = \frac{1}{2}e^{-t/2}$ and the field is

$$E(\mathbf{x}) = \lambda \int\limits_0^{|\xi|^2} \frac{dt\, e^{-t/2}}{\sqrt{\bar{z}^2 - t(\sigma_x^2 - \sigma_y^2)}} .$$

Now making the change of integration variable $2(\sigma_x^2 - \sigma_y^2)s^2 = \bar{z}^2 - t(\sigma_x^2 - \sigma_y^2)$ and using the definition of the complex error function [9] $w(z)$,

$$\int\limits_0^z ds\, e^{s^2} = \frac{\sqrt{\pi}}{2i}\left[e^{z^2} w(z) - 1\right]$$

gives

$$E(\mathbf{x}) = i\lambda\sqrt{\frac{2\pi}{\sigma_x^2 - \sigma_y^2}}\, e^{-s_1^2}\left[e^{s_2^2}\, w(s_2) - e^{s_1^2}\, w(s_1)\right]$$

where $s_1$ and $s_2$ are

$$s_1 \equiv \frac{\bar{z}}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}}$$

$$s_2 \equiv \sqrt{\frac{\bar{z}^2 - |\xi|^2(\sigma_x^2 - \sigma_y^2)}{2(\sigma_x^2 - \sigma_y^2)}} = \frac{\bar{\omega}}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} .$$

Substituting these and using the identity (19), the electric field becomes

$$E(\mathbf{x}) = i\lambda\sqrt{\frac{2\pi}{\sigma_x^2 - \sigma_y^2}}\left[e^{-|\xi|^2/2}\, w\left(\frac{\bar{\omega}}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}}\right) - w\left(\frac{\bar{z}}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}}\right)\right]$$

which agrees with the known result [10].

### E. Distributions that are polynomials in $t$.

By using the elementary recursion relation

$$\int \frac{dt\, t^n}{\sqrt{1-kt}} = \frac{-2}{(2n+1)k}\left[t^n\sqrt{1-kt} - n\int \frac{dt\, t^{n-1}}{\sqrt{1-kt}}\right] \qquad (26)$$

where $k \equiv (a^2 - b^2)/\bar{z}^2$, it is obviously possible to compute the complex electric field for a charge density that is an arbitrary polynomial function of $t$. However, straightforward application of Eq. (26) yields an expression with powers of $a^2 - b^2$ in the denominator. If one wishes to obtain a result with a manifestly regular round-beam limit, there is a systematic way to do so by using the definition of the hypergeometric function followed by a quadratic transformation [11], thus

$$\begin{aligned}
\int\limits_0^T \frac{dt\, t^n}{\sqrt{1-kt}} &= T^{n+1}\int\limits_0^1 \frac{ds\, s^n}{\sqrt{1-kTs}} \\
&= \frac{T^{n+1}}{n+1}\, {}_2F_1\left(\tfrac{1}{2}, n+1, n+2; kT\right) \\
&= \frac{T^{n+1}}{n+1}\left(\frac{2}{1+\sqrt{1-kT}}\right)^{n+1} {}_2F_1\left(-n, n+1, n+2; (1-\sqrt{1-kT})/2\right)
\end{aligned}$$

where $T$ is the appropriate top limit of the integral. For integer values of $n$, the hypergeometric function in the last equation is a polynomial of degree $n$ in the variable $(1-\sqrt{1-kT})/2$ and hence this result is in the form of Eq. (24b).

As a simple example, we consider the density [7, 8]

$$\rho(\mathbf{x}) = \begin{cases} \dfrac{2\lambda}{\pi ab}\left(1 - x^2/a^2 - y^2/b^2\right) & \text{if } x^2/a^2 + y^2/b^2 \le 1 \\ 0 & \text{if } x^2/a^2 + y^2/b^2 > 1\,. \end{cases}$$

The normalized density reads $\hat{\rho}(t) = 2(1-t)\theta(1-t)$, and the field is computed following the same steps as in the uniform-charge-distribution case. The result is

$$E(\mathbf{x}) = \begin{cases} \dfrac{8\lambda\xi}{a+b}\left(1 - \dfrac{(2\bar{z}+\bar{\omega})\xi}{3(a+b)}\right) & \text{if } x^2/a^2 + y^2/b^2 \le 1 \\ \dfrac{2\lambda}{\bar{z}}\left(\dfrac{2\bar{z}}{\bar{z}+\sqrt{\bar{z}^2-a^2+b^2}}\right)^2 \cdot \left(\dfrac{\bar{z}+2\sqrt{\bar{z}^2-a^2+b^2}}{3\bar{z}}\right) & \text{if } x^2/a^2 + y^2/b^2 > 1. \end{cases}$$

These expressions are in agreement with the short- and long-distance limits (21) and (22). In the region with charge, the field is a polynomial function of $x$ and $y$ and hence it is manifestly unambiguous and devoid of discontinuities; it also agrees with the known result [7]. In the charge-free region, the field obeys the analyticity property (2). A certain amount of gymnastics with the identities (18) shows that the field is continuous at the edge of the distribution.

### F. Other distributions.

Eq. (17) also yields closed expressions for many other kinds of densities. For example, for infinite-extent densities of the form $\hat{\rho}(t) = (1+t)^{-n/2}$, one can compute the field for any integer $n > 1$ by using a recursion relation that expresses the integral in terms of the density for $n = 1$ or $n = 2$, which can be found in a table of integrals.[4] The same technique applies to densities of the form $t^n e^{-t/2}$, to finite-extent densities of the form $\theta(1-t)\cdot(1-t)^{n/2}$, and to polynomials in $\sqrt{t}$.

For more complicated distributions, it is in principle possible to combine the information from the short- and long-distance expansions, Eqs. (21) and (22), to find approximate analytic forms for the field [12, 13].

## V. REMARKS.

### A. Form factor.

If a factor $\bar{z}$ is pulled outside the square root in Eq. (20), the field is written in the form

$$E(\mathbf{x}) = E_0(\mathbf{x}) \cdot F(\mathbf{x})$$

where $F$ is a dimensionless "form factor" that describes the finite-size effects,

$$F(\mathbf{x}) = \int_0^{|\xi|^2} \frac{dt\,\hat{\rho}(t)}{\sqrt{1 - t(a^2-b^2)/\bar{z}^2}}\,.$$

With the cut structure discussed earlier, it can be easily shown that the square root in this expression is an even-parity function, and hence so is $F(\mathbf{x})$ (the line-charge field $E_0(\mathbf{x})$, of course, is of odd parity, making the overall field $E(\mathbf{x})$ of odd parity). Since the field is well defined for all $z$, so is the form factor. The leading behaviors shown in Eqs. (21) and (22) translate into

$$F(\mathbf{x}) \to \begin{cases} \dfrac{2\lambda\hat{\rho}(0)}{a+b}\,\xi\bar{z} & \text{as } |z| \to 0 \\ 1 & \text{as } |z| \to \infty\,. \end{cases}$$

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[4] Although the integral in Eq. (20) can be done for any integer $n$, it should be noted that the result for the case $n \le 2$ is suspect because $\rho$ is not integrable; see footnote 2. The case $n = 4$ is discussed in Ref. [5].

### B. Computational issues.

The ANSI-standard definition of the Fortran function CSQRT(Z) evaluates the square root $(z^2 - a^2 + b^2)^{1/2}$ incorrectly for our purposes: CSQRT(Z) turns it into an even-parity function, corresponding to the cuts in Fig. 1b. By the same token, CSQRT(Z) also makes $[1-(a^2-b^2)/z^2]^{1/2}$ an even-parity function, which is correct for our purposes. Therefore, if one wants to carry out computations in standard Fortran without conditioning the calculation to the quadrant to which $z$ belongs, it is simplest to code the formulas by first making the replacement $(z^2 - a^2 + b^2)^{1/2} \rightarrow z[1-(a^2-b^2)/z^2]^{1/2}$.

Although the complex expressions for the field are simpler than those for the real and imaginary parts, they are slower to compute because, typically, compilers are not optimized for complex arithmetic. Results from a benchmark with double-precision arithmetic on a VAX 6610 computer (VMS Fortran v. 6.1) show that evaluating Eq. (25) (plus the $y$-component) is faster by a factor of 2 than evaluating Eq. (24a) and then taking the real and imaginary part. In single precision, the corresponding factor is 2.5. However, for more complicated cases, we expect that the complex expressions are more competitive from the perspective of computational speed due to their much simpler form.

### C. Distribution functions with contours other than elliptical.

The method described can be extended, in principle, to charge distributions that depend on $x$ and $y$ only through a positive-definite combination $c(x, y)$ such that

$$c(x, y) = t$$

represents a simple (i.e., nonintersecting), smooth, closed curve. However, it seems clear that the elliptical contours are the simplest.

## VI. CONCLUSIONS.

We have presented a formalism that yields simple, compact expressions for the electric field of two-dimensional charge distributions with elliptical contours. Cauchy's theorem plays a central role in the calculation. We have reproduced in closed form the known results for the cases of the uniformly charged ellipse, the Gaussian distribution, and one example of a polynomial distribution. Our formalism allows simplification of analytical work and of coding in numerical applications in electromagnetism or in gravitational theory involving elliptical distributions. It is in principle possible to extend the method to distributions whose contours are more complicated nonintersecting closed curves. However, the complex-number nature of the method restricts it to effectively two-dimensional applications.

## VII. ACKNOWLEDGMENTS.

I am indebted to J. D. Jackson for many valuable comments. I am grateful to L. Teng, L. Smith and R. Baartman for correspondence on their work and for comments, and to K. J. Kim for encouragement.

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