LHC Project Note 97

1 July 1997

Scott.Berg@cern.ch

Energy Gain in an Electron Cloud During the Passage of a Bunch

J. Scott Berg

Keywords: Cryogenic Heat Load, Electron Cloud, Ohmi Effect

Summary

When a proton bunch passes by, electrons which are in the chamber will gain some energy due to the passage of this bunch. This report presents approximate formulas, valid in the absence of an external magnetic field, for the total energy gained by a test charge from a relativistic beam as a function of the initial radial position of the particle. These formulas are used to estimate the total energy gained by a uniform distribution of test charges. One can thus obtain an approximate scaling of the energy deposition on the beam screen with various parameters in the problem. These results are compared to numerical integration, demonstrating that the longitudinal bunch shape has a significant effect on the energy gained by a test charge. The assumption used in the analytic estimate that the longitudinal bunch shape is rectangular significantly overestimates the maximum energy gained as well as the energy gained by particles at low radii. Nonetheless, the analytic estimate gives relatively accurate results for the average energy gained when the test charges are uniformly distributed in the chamber. The paper also gives a computation of the distribution in energy gain of the initially produced photoelectrons, which happens to be independent of the longitudinal bunch distribution. Results are given for the parameters for the LHC. Finally, comments on the step size necessary in a simulation are made.

1 Introduction

Synchrotron radiation from proton bunches in the LHC will create photoelectrons in the beam pipe. These photoelectrons will be pulled toward the oppositely charged proton beam when it passes by. When they hit the opposite wall, these photoelectrons generate secondary electrons which can in turn be accelerated by the next bunch. For moderate values of the secondary emission yield, this mechanism leads to the build-up of an electron cloud almost uniformly distributed in the beam pipe. This process not only can potentially cause an instability [1, 2, 3], but the energy transferred from the beam to the electrons will eventually be deposited in the beam screen, creating an additional heat load on the cryogenic system [4]. For both of these reasons, it is important to have an idea of the energy transferred from the beam to the electron cloud.

The energy transfer is simple to analyze if the electron is far away from the beam. In this case, the electron is essentially stationary as a bunch passes, and the bunch can be treated as simply changing the momentum of the electron by an amount which only depends on its radial position in the beam pipe. This is often the situation for the photoelectrons when they are first produced. The distribution of the energies of the photoelectrons after the first bunch can then be computed relatively accurately using this approximation.

However, particles which are closer to the beam can get temporarily trapped in the radial potential of the bunch, and will thus oscillate around it. The energy transfer to such particles is more difficult to analyze. This difficulty can be partially overcome by assuming that the longitudinal distribution in the bunch is rectangular. The problem then becomes an autonomous one-dimensional problem (when certain other approximations are made). While the solution can still not be written down in terms of elementary functions, certain approximations can be made to get an upper bound on the energy transferred to the electron.

A non-autonomous version of the one-dimensional problem, where the longitudinal distribution in the bunch is non-rectangular, can be treated easily by numerically integrating the equations of motion. The results are quantitatively and qualitatively different from what one obtains with the rectangular bunch.

This report first computes an analytic estimate of the energy gained by a test charge as a function of its initial position, assuming that it has no initial momentum. It then uses this result to obtain an estimate of the energy transferred to a cloud of test charges uniformly distributed in the beam pipe. One can then determine how the energy deposition varies with beam current. Also, the distribution of the kick received by the initially produced photoelectrons is computed, as is the maximum energy they receive. Results are given for the parameters in the LHC.

The report then describes a numerical integration of the equations of motion which takes into account the longitudinal bunch distribution. The simulation is run with the LHC parameters, and the results are compared with the analytic estimates. The effect of the bunch shape is also examined. Comments are made on the step size that is necessary to correctly do a simulation of these effects.

2 Equations Describing the Particle's Motion

2.1 The Potential

Consider a beam pipe with translational symmetry in one direction (the $\hat{z}$ direction) with a highly relativistic source distribution moving in the $\hat{z}$ direction. Furthermore, assume that the beam pipe consists of a single perfectly conducting surface. The vector and scalar electromagnetic potentials due to the distribution can be written as a power series in $1/\gamma^2 = (1 - v^2/c^2)$, v being the source distribution's velocity. They are given by $A_z = \beta\phi$ and $\boldsymbol{A}_\perp = \boldsymbol{0}$, where ϕ is the scalar potential, $\boldsymbol{A}$ is the vector potential, β is the velocity divided by c, and the subscript $\perp$ refers to the part of the vector perpendicular to $\hat{z}$. To lowest order in $1/\gamma^2$,

$$\nabla_\perp^2 \phi = -4\pi\rho,$$

where ρ is the charge density in the beam. Note that the longitudinal position z is really just a parameter in this equation, since the Laplacian operator is only the transverse one.

For a cylindrically symmetric beam pipe and a cylindrically symmetric source distribution ρ, ϕ is given by

$$\phi(r,z) = 4\pi \int_0^r r' \ln \frac{r'}{r} \rho(r', z) dr'.$$

Note that ϕ has been written so that $\phi(0, z) = 0$.

2.2 The Hamiltonian

Consider a non-relativistic test charge with mass m_T and charge $q_T = Q_T e$. The Hamiltonian describing its motion is

$$H = \frac{p_r^2}{2m_T} + \frac{p_\theta^2}{2r^2 m_T} + \frac{\left[p_z - \frac{q_T \beta}{c}\phi(r, z - \beta ct)\right]^2}{2m_T} + q_T \phi(r, z - \beta ct). \tag{1}$$

2.3 Simplifying Assumptions

Since the test charges are non-relativistic, and the transverse electric and magnetic fields due to the source distribution are comparable, one expects that magnetic fields will only have a small effect on the beam. Thus, one can drop the A_z term in the Hamiltonian (1).

The bunch length is large compared to the transverse bunch size, and it is generally large compared to the beam pipe radius. Thus, the energy gained in the longitudinal direction is relatively small. This allows one to drop the p_z term altogether, and z is simply a parameter in the problem. Note that nonetheless, there is an effect which is potentially interesting here. The test charge can potentially receive a longitudinal kick due to the passage of the bunch. This longitudinal kick is being ignored here. To treat it properly, the entire p_z term should be kept, including A_z. There will probably be little effect on the total energy gain, but knowledge of how particles drift longitudinally between bunch passages is potentially important.

Finally, the angular momentum p_θ of the test charge is taken to be zero.

The simplified Hamiltonian is thus

$$H = \frac{p_r^2}{2m_T} + q_T \phi(r, z - \beta ct).$$

2.4 Scaled Problem

It is convenient to work in terms of scaled variables. First of all, write the source charge density for a bunch of N_S particles with charge $q_S = Q_S e$ as

$$\rho(r,z) = \frac{q_S N_S}{2\pi \sigma_\perp^2 \sigma_\ell} f\left(\frac{r}{\sigma_\perp}\right) \lambda\left(\frac{z}{\sigma_\ell}\right),$$

where

$$\int_0^\infty u\, f(u) du = 1 \qquad \int_0^\infty u^3 f(u) du = 1$$

$$\int_{-\infty}^\infty \lambda(u) du = 1 \qquad \int_{-\infty}^\infty u^2 \lambda(u) du = 1.$$

f and λ are normalized radial and longitudinal bunch distributions respectively, and thus $\sigma_\perp$ is the r.m.s. beam radius, and σ_ℓ is the r.m.s. bunch length. Next, change variables as follows:

$$\bar{r} = \frac{r}{\sigma_\perp} \qquad \bar{z} = \frac{z}{\sigma_\ell} \qquad \bar{t} = \frac{\beta c t}{\sigma_\ell}$$
$$\bar{p}_r = \frac{\sigma_\ell p_r}{m_T \beta c \sigma_\perp} \qquad \bar{p}_z = \frac{p_z \sigma_\ell^2}{m_T \beta c \sigma_\perp^2} \qquad \bar{H} = \frac{\sigma_\ell^2 H}{m_T \beta^2 c^2 \sigma_\perp^2}.$$

Note that this transformation is not symplectic; nonetheless, it preserves Hamilton's equations of motion because it is a composition of a symplectic transformation and a simultaneous scaling of the momenta and the Hamiltonian by a constant factor. The scaled Hamiltonian in these scaled variables becomes

$$\bar{H} = \frac{\bar{p}_r^2}{2} + \frac{Q_S Q_T N_S r_T \sigma_\ell}{\beta^2 \sigma_\perp^2} \bar{\phi}(\bar{r}) \lambda(\bar{z} - \bar{t}), \tag{2}$$

where $r_T = e^2/m_T c^2$ and

$$\bar{\phi}(u) = 2 \int_0^u u' \ln \frac{u'}{u} f(u') du'$$

3 Approximate Analytic Solution

Assume that Q_S and Q_T have different signs, so that the test charges are pulled toward the source distribution.

One expects two different types of motion to occur during the passage of the source distribution. The first is if the source distribution passes by before the test charge has started to move very much. This will be called the kick approximation. The second case is when the test charge is so strongly pulled by the source distribution that it begins to make oscillations around the source distribution. This case can be treated approximately by assuming that the charge density of the source distribution is independent of time (this assumption is exact for a bunch whose longitudinal shape is rectangular). This will be called the autonomous approximation.

For computing the energy gained by a test charge, it is always assumed that the charge has no initial momentum.

3.1 Kick Approximation

When the test charge's position doesn't change, the total change in $\bar{p}_r$ as a result of the bunch passage is

$$\Delta \bar{p}_r = -\frac{Q_S Q_T N_S r_T \sigma_\ell}{\beta^2 \sigma_\perp^2} \frac{d\bar{\phi}}{d\bar{r}},$$

giving a net energy gain of

$$\Delta E_{\text{kick}} = \frac{1}{2} m_T c^2 \left(\frac{Q_S Q_T N_S r_T}{\beta \sigma_\perp} \right)^2 \left(\frac{d\bar{\phi}}{d\bar{r}} \right)^2, \tag{3}$$

which is independent of the longitudinal bunch distribution.

In the case where the test charges are photoelectrons produced by the beam itself, an interesting effect occurs. Because the beam is moving close to the speed of light, if a photon is produced by a slice of the beam at a position z within the bunch, that photon will hit the beam pipe wall at that same position z in the bunch as the bunch passes by. This assumes that there is no loss of synchronism between the bunch and the photons, which should be true as long as the photons don't go through too many reflections. Thus, first of all, the initial position of the photoelectron will be at the beam pipe wall, $r = b$. The beam pipe wall is generally well outside of the bunch. Well outside the beam core,

$$\bar{\phi}(\bar{r}) = -2\ln\frac{\bar{r}}{c_0} \tag{4}$$

$$\ln c_0 = \int_0^\infty u\ln u\, f(u)du, \tag{5}$$

where $c_0 = \sqrt{2}e^{-\gamma/2} \approx 1.05968$ for a transverse Gaussian distribution (γ is Euler's constant). Hence, if it had seen the kick from the entire bunch, it would have received an energy gain ΔE_{kick} of

$$\Delta E_{\text{wall}} = 2m_T c^2 \left(\frac{Q_S Q_T N_S r_T}{\beta b}\right)^2. \tag{6}$$

However, the photoelectron which is produced will only see the portion of the bunch which follows z. Thus, in the kick approximation, the momentum gained by a photoelectron emitted at scaled position $\bar{z}$ will be

$$P = \Delta p_{\text{wall}} \int_{-\infty}^{\bar{z}} \lambda(z')dz', \tag{7}$$

where $\Delta p_{\text{wall}} = \sqrt{2m_T \Delta E_{\text{wall}}}$, assuming that positive $\bar{z}$ refers to the head of the bunch. Since the number of photoelectrons produced at scaled longitudinal position $\bar{z}$ is proportional to $\lambda(\bar{z})$, the momentum gained by the photoelectrons will be uniformly distributed between 0 and Δp_{wall}, the energy gain E will be distributed from 0 to ΔE_{wall} according to the distribution $1/(2\sqrt{E\Delta E_{\text{wall}}})$, and the average energy gained will be $\Delta E_{\text{wall}}/3$, as demonstrated in appendix A (this results was also demonstrated by S. A. Heifets in [2]). All this is independent of the actual longitudinal distribution $\lambda(z)$. Note that this paragraph only applies to the energy gained by the photoelectrons due to the *first* bunch passing by after they are produced. Also, this is valid as long as the kick approximation holds near the beam pipe wall. Below it will be demonstrated that this will be true in many cases, including the LHC parameters examined here.

3.2 Autonomous Approximation

This approximation ignores the time-dependence in λ during the passage of the bunch. In this case, the test charge executes circular-like motion in phase space. This will be valid if the frequency of this oscillation is large compared to the time scales over which λ changes, and (usually contrastingly!) when λ goes from zero to nonzero over a time scale that is small compared to this oscillation time. In this case, ignoring any initial momentum of the test charge, the maximum energy gained by the particle will be given by its initial potential energy

$$\Delta E_{\mathrm{autonomous}} = m_T c^2 \frac{Q_S Q_T N_S r_T}{\sigma_\ell} \bar{\phi}(\bar{r}) \lambda(\bar{z} - \bar{t}). \tag{8}$$

The actual energy gained depends on the exact period of the oscillation and the length of time that the particle is subject to the force. As an initial estimate, one can assume the particles to have half of this energy, the reason being that the average value of $\bar{p}_r^2/2$ is half of its maximum value for a harmonic oscillator whose oscillation is stopped at a random time. In practice, that "random time" comes from the fact that the period varies with the initial radius of the test charge, and so we're averaging over the period, and not the time. For amplitudes where the potential does not resemble that of a harmonic oscillator, one expects this fraction to be different from 1/2. The fraction should be less, since the potential is proportional to $\ln r$ for large radii, which increases less rapidly than r^2.

There is finally the question of which value of λ to use. The energy gained depends on the position at which you choose to look at λ; this is why the $\bar{z}$ and $\bar{t}$ dependence was left in (8). To make a pessimistic assumption, one should take the peak value, denoted $\lambda_{\max}$. The idea is that the maximum potential energy seen may determine the energy gained by the particle. In the case of a longitudinally Gaussian distribution, $\lambda_{\max} = 1/\sqrt{2\pi}$.

Finally, in practice, the energy gained will be less than what is discussed above due to the variation of λ with time. If the variation of λ is slow compared to the oscillation period, one expects an adiabatic-invariance type of behavior [5] which could lead to very little energy being gained.

Thus, these assumptions tend to lead to over-estimates of the energy that will be gained.

3.3 Deciding Which Approximation to Use

The oscillation period in the autonomous approximation gives an idea of when to use the autonomous approximation and when to use the kick approximation. If the oscillation period is long compared to the time for the source distribution to pass, then the kick approximation should work well. Otherwise, one should use the autonomous approximation.

At large radii, $\bar{\phi}$ approaches $-2\ln r$, which increases more slowly than r^2. Thus, one expects longer oscillation periods at large radii. There will therefore be a transition radius r_C above which the kick approximation is used, and below which the autonomous approximation is used. The maximum energy is gained in the autonomous approximation if the test charge executes a quarter oscillation. Thus, it is sensible to consider r_C to be the radius for which the time for the bunch to pass is equal to a quarter oscillation period.

A simplified model to determine the time for the bunch to pass is to take λ to be a step function whose value is equal to the $\lambda_{\max}$, and whose length is such that the integral of λ remains 1. Thus, it's length will be $\sigma_\ell/\lambda_{\max}$ ($1/\lambda_{\max}$ in dimensionless units).

The time $\Delta\bar{t}$ for one complete oscillation can thus be computed to be

$$\Delta\bar{t} = 4\left(-\frac{Q_S Q_T N_S r_T \sigma_\ell}{\beta^2 \sigma_\perp^2}\lambda_{\max}\right)^{-1/2} \int_0^{\bar{r}} \frac{dr'}{\sqrt{2[\bar{\phi}(r') - \bar{\phi}(\bar{r})]}}. \tag{9}$$

The aforementioned approximations mean that $\Delta\bar{t} = 4/\lambda_{\max}$ when $\bar{r} = r_C/\sigma_\perp$. Thus, r_C is found from the solution of the implicit equation

$$\int_0^{r_C/\sigma_\perp} \frac{dr'}{\sqrt{2[\bar{\phi}(r') - \bar{\phi}(r_C/\sigma_\perp)]}} = \sqrt{-\frac{Q_S Q_T N_S r_T \sigma_\ell}{\beta^2 \sigma_\perp^2 \lambda_{\max}}}. \tag{10}$$

Equation (4) allows the left hand side of (10) to be evaluated approximately to be $r_C\sqrt{\pi}/2\sigma_\perp$, using the fact that

$$\int_0^1 \frac{dx}{\sqrt{-\ln x}} = \sqrt{\pi}.$$

Thus, the transition radius is

$$r_C = 2\sqrt{-\frac{Q_S Q_T N_S r_T \sigma_\ell}{\pi \beta^2 \lambda_{\max}}}. \tag{11}$$

The approximations used here indicate that this only makes sense when $r_C \gg \sigma_\perp$.

An estimate of the maximum energy gained by any test particle can be obtained (fairly accurately in practice) by using $\bar{r} = r_C/\sigma_\perp$ in equation (8). Using the approximation (4) for the potential, the maximum energy gained will be approximately

$$\Delta E_{\max} = -2 m_T c^2 \frac{Q_S Q_T N_S r_T}{\sigma_\ell} \lambda_{\max} \ln \frac{r_C}{c_0 \sigma_\perp}. \tag{12}$$

3.4 The Total Energy Deposition

Assume that just before the bunch arrives, there is a uniform distribution of N_T test charges in a circular beam pipe of radius b (the radial distribution function is $2r/b^2$). Ignoring the contribution from small radii within the beam core, the total energy gained by all the test charges in the autonomous approximation (from within the radius r_C) is

$$\overline{\Delta E_{\text{autonomous}}} = N_T m_T c^2 \left(\frac{Q_S Q_T N_S r_T}{\beta b}\right)^2 \frac{4}{\pi} \left(\ln \frac{r_C}{c_0 \sigma_\perp} - \frac{1}{2}\right). \tag{13}$$

The contribution to the total energy from those particles governed by the kick approximation, outside the radius r_C, is

$$\overline{\Delta E_{\text{kick}}} = N_T m_T c^2 \left(\frac{Q_S Q_T N_S r_T}{\beta b}\right)^2 4 \ln \frac{b}{r_C}. \tag{14}$$

In both cases, the integrations were done assuming (4). This gets any contributions from within the source distribution itself wrong, but these contributions should be negligible, both because there are few test charges at this radius, and because the energy gained by those particles is small.

Notice that the overall dependence for both the autonomous approximation and the kick approximation is the same. The logarithm factors containing r_C depend very weakly on the parameters. As long as r_C stays away from the beam pipe wall and the source distribution, one expects these expressions to approximate the total energy gain fairly well (subject to all aforementioned conditions).

Thus, in particular, for a fixed number of test charges N_T, we expect the energy deposition to increase with the square of the beam current, and be roughly independent of the bunch length and transverse beam size. This holds true as long as r_C stays well outside the bunch yet within the beam pipe. There is also an inverse square dependence of the energy deposition on b.

In the case of photoelectrons being produced by the beam, the number of photoelectrons per unit length scales linearly with the beam current. Thus, the overall dependence of the specific energy deposition is with the cube of the beam current.

If r_C becomes larger than the beam pipe radius (which will happen for very large currents), all particles should be treated with the autonomous approximation, giving an energy deposition of

$$-N_T m_T c^2 \frac{Q_S Q_T N_S r_T \lambda_{\max}}{\sigma_\ell}\left(\ln\frac{b}{c_0\sigma_\perp} - \frac{1}{2}\right).$$

Note that this increases only linearly with current, and now it also decreases inversely with bunch length.

If r_C becomes comparable to or less than $\sigma_\perp$ (very low currents), then all particles should be treated with the kick approximation. A good approximation would probably be to take $r_C = c_0 e^{-1/2}\sigma_\perp$ and use (14) (the choice of r_C coming from making (13) zero). The exact value for r_C won't matter too much due to the logarithmic dependence.

3.5 Approximate Analytic Results for LHC

The relevant parameters for the LHC at top energy are

	N_S	σ_ℓ	$\sigma_\perp$	b
Nominal	1.05×10^{11}	7.7 cm	0.2 mm	2.2 cm
Ultimate	1.6×10^{11}	7.7 cm	0.2 mm	2.2 cm

Here b is taken to be the maximum aperture of the beam screen (optimistic assumption). Using equations (6), (11), (12), (13), and (14), the results are

	r_C	$\overline{\Delta E_{\text{autonomous}}}/N_T$	$\overline{\Delta E_{\text{kick}}}/N_T$	$\Delta E_{\text{total}}/N_T$	$\Delta E_{\max}$	ΔE_{wall}
Nominal	0.85 cm	376 eV	350 eV	726 eV	5789 eV	189 eV
Ultimate	1.05 cm	931 eV	632 eV	1563 eV	9324 eV	439 eV

Note that r_C is well outside the beam yet well within the beam pipe, as it must be for the above computations to be valid. Also, the energy gained per electron is much less than the rest mass energy of the electron, thus the non-relativistic approximation holds. Finally, note that, as described in section 3.1, for the bunch passage in which the photoelectrons are generated, the average energy gained is $\Delta E_{\text{wall}}/3$, which is 63 eV for the nominal LHC parameters and 146 eV for the ultimate parameters.

4 Numerical Computation

Using symplectic integration, one can determine the motion for a particle governed by the Hamiltonian (2). In fact, one can easily take into account the longitudinal bunch shape λ, and see if that has a significant effect. One can then find the energy gained by the particles as a function of radius, numerically integrate this over a uniform transverse distribution of test charges, and compare the results to those which where obtained with the above approximate analytic solution.

4.1 Step Sizes

To begin, one must determine the step size to be used in the integration, both in time and in radius. If the time step is too large, the results can be very inaccurate. To obtain these estimates, use the autonomous approximation described above. The function $\bar{\phi}(u)$ can be expanded in a Taylor series in u about the origin assuming that one has a Taylor series for f about the origin:

$$\bar{\phi}(u) = -2 \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(k+2)^2} f^{(k)}(0) u^{k+2}. \tag{15}$$

Assuming f to be a function of r^2, and taking the two lowest order terms in (15), the integral in (9) can be computed to lowest order in $\bar{r}$, giving

$$\Delta\bar{t} = 2\pi \left(-\frac{Q_S Q_T N_S r_T \sigma_\ell}{\beta^2 \sigma_\perp^2} \lambda_{\max} \right)^{-1/2} f(0) \left[1 - \frac{3}{32} \frac{f''(0)}{f(0)} \bar{r}^2 \right]. \tag{16}$$

Note that $f(0) > 0$, while usually $f''(0) < 0$. For a transverse Gaussian distribution, $f(0) = 1$ and $f''(0) = -1$. Note that $\Delta\bar{t} = 1/[\lambda_{\max} n_O(\bar{r})]$, where $n_O(\bar{r})$ is the number of oscillations executed during the bunch passage by a particle starting at $\bar{r}$.

Since the shortest oscillation period is that for particles near $\bar{r} = 0$, the time step should be chosen to be a small fraction (f_t) of

$$\Delta\bar{t}_{\text{step}} = 2\pi \left(-\frac{Q_S Q_T N_S r_T \sigma_\ell}{\beta^2 \sigma_\perp^2} \lambda_{\max} \right)^{-1/2} f(0). \tag{17}$$

Note that choosing f_t larger than or comparable to 1 will probably result in very inaccurate results! This may in fact be the reason for the excessively high peak in electron energy seen in figure 6 of [3] (since the kick approximation, which is effectively what a single integration step does, will vastly overestimate the energy transferred to a particle at low radii).

The radial step required to compute average values of the energy gain in the electron cloud is more difficult to determine. Ideally, one wants to determine the difference in $\bar{r}$ which corresponds to a change in n_O of 1. For small $\bar{r}$, using equation (16), one finds

$$\frac{n_O(0)}{n_O(\bar{r})} = 1 - \frac{3}{32} \frac{f''(0)}{f(0)} \bar{r}^2.$$

One can compute $d\bar{r}/dn_O$, finding

$$\frac{d\bar{r}}{dn_O(\bar{r})} = -\sqrt{-\frac{8}{3} \frac{f(0)}{f''(0)} \frac{n_O(0)}{n_O^2(\bar{r})}} \left(\frac{n_O(0)}{n_O(\bar{r})} - 1 \right)^{-1/2}.$$

Computing the second derivative of this with respect to $n_O(\bar{r})$ shows that the minimum of the absolute value of this occurs for $n_O(\bar{r}) = \frac{3}{4}n_O(0)$. Thus, the minimum value of $|d\bar{r}/dn_O|$ is

$$-\Delta\bar{r}_{\text{step}} = -\frac{32}{9}\sqrt{-2\frac{f(0)}{f''(0)}}\frac{1}{n_O(0)} = -\frac{64\pi}{9}\sqrt{-2\frac{f(0)}{f''(0)}}\left(-\frac{Q_S Q_T N_S r_T \sigma_\ell}{\beta^2\sigma_\perp^2\lambda_{\max}}\right)^{-1/2} f(0). \quad (18)$$

Thus, the radial integration step should be a small fraction (f_r) of this quantity. In most cases this will underestimate the radial step size, since the expansion (16) becomes invalid once $\bar{r}$ becomes very large.

4.2 Numerical Results for LHC

A fourth-order symplectic integrator [6] is used to integrate the Hamiltonian (2), assuming that initially $\bar{p}_r = 0$ and $\bar{r}$ is a given initial condition. The step size is governed by the considerations given in the previous section. The fractions f_t and f_r are chosen to be 0.003 and 0.03 respectively, but the values 0.03 and 0.1 are generally sufficient to get reasonable results (for larger time steps one begins to see chaotic behavior). The values for Δt_{step}, Δr_{step}, actual step sizes for "reasonable results," and n_{rect}, a reasonable number of time steps for a rectangular bunch, are

	$\Delta\bar{t}_{\text{step}}$	$\Delta\bar{r}_{\text{step}}$	$0.03\Delta\bar{t}_{\text{step}}$	$0.1\Delta\bar{r}_{\text{step}}$	n_{rect}
Nominal	0.417	0.836	0.0125	0.0836	200
Ultimate	0.338	0.677	0.0101	0.0677	247

Three different forms for the longitudinal bunch distribution λ are used:

$$\begin{aligned}
\lambda_{\text{rect}}(\bar{z}) &= \frac{1}{\sqrt{2\pi}} & |\bar{z}| &< \sqrt{\frac{\pi}{2}} \\
\lambda_{\text{gauss}}(\bar{z}) &= \frac{1}{\sqrt{2\pi}}e^{-\bar{z}^2/2} & & \\
\lambda_{\text{parab}}(\bar{z}) &= \frac{35}{96}\left(1-\frac{z^2}{9}\right)^3 & |\bar{z}| &< 3
\end{aligned}$$

λ_{rect} is what we assume for the approximate analytic solution in section 3. λ_{parab} is a good approximation for the distribution in a proton machine (it's similar to a Gaussian distribution, but truncated smoothly at $3\sigma_\ell$), whereas λ_{gauss} is probably more appropriate for electron machines. In all cases, the radial bunch distribution $f(r)$ is assumed to be a Gaussian.

The resulting energy gain as a function of radius is shown in figures 1 and 2. Also shown are the energy gains in the autonomous approximation as well as in the kick approximation. The average energy gain for test charges uniformly distributed in the vacuum chamber can be found by integrating the energy gain as a function of r times $2r/b^2$. The resulting average energy gains are

	Rectangular	Gaussian	Parabolic
Nominal	712 eV	644 eV	638 eV
Ultimate	1454 eV	1287 eV	1274 eV

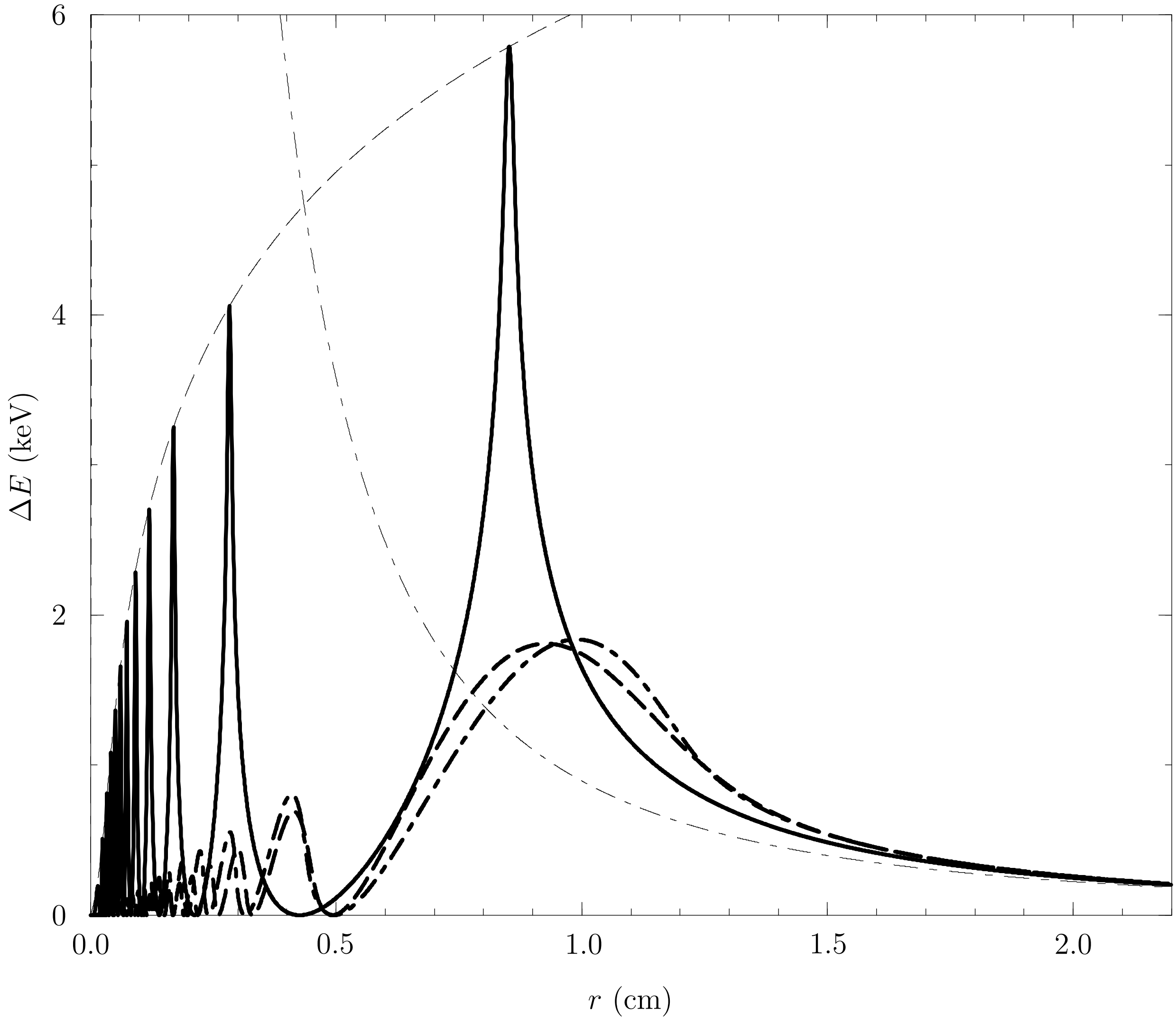

Figure 1: Energy gain as a function of initial particle radius for the nominal LHC parameters. The solid line is for a rectangular bunch, the dashed line is for a Gaussian bunch, and the dot-dashed line is for a parabolic-like bunch. The thin dashed line shows what the maximum energy gain is expected to be in the autonomous approximation, whereas the thin dot-dashed line is the energy gain expected in the kick approximation.

Comparing the numbers found in section 3.5, the energy gains for the rectangular distribution compare relatively well to the analytic estimates. Examining figures 1 and 2 sheds some light on this. At large radii, the kick approximation gives a good approximation to the energy gain, and for low radii, the maximum energy in the peaks is exactly the maximum energy expected in the autonomous approximation. The tradeoff between the estimates being lower for large radii and higher for low radii (because the average energy gain is less than half of the maximum, as can be seen from the shape of the peaks) keeps the analytic estimates relatively close to the computed values. Note that what is computed here agrees qualitatively

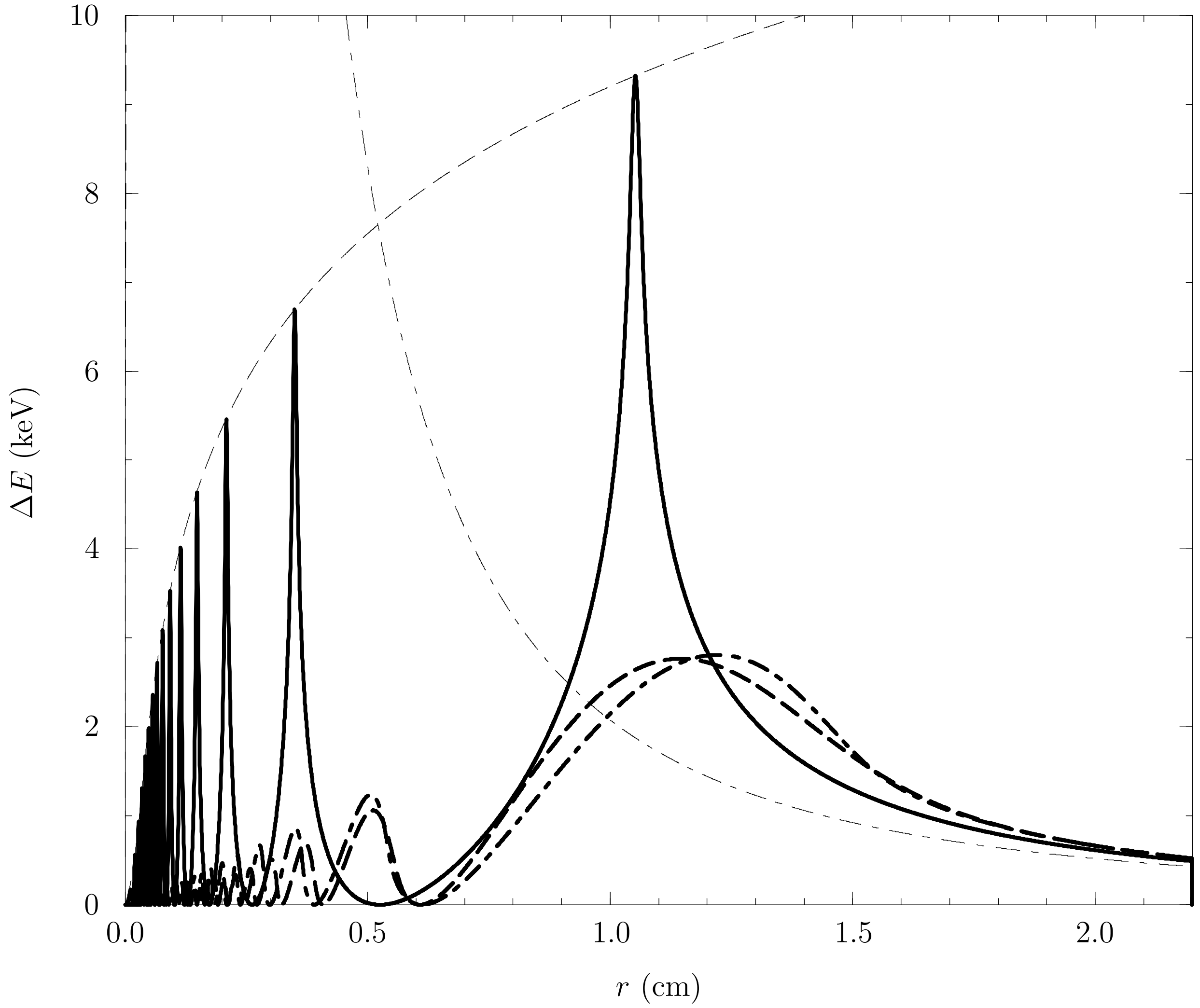

Figure 2: Energy gain as a function of initial particle radius for the ultimate LHC parameters. See figure 1 for the meaning of the lines.

with what is found in [4], in particular figure 1.

The values for r_C and $\Delta E_{\max}$ found in section 3.5 compare well with the location and height of the largest peak in figures 1 and 2. In fact, using equation (9) and the fact that the integral in that formula is proportional to $\bar{r}$ (see the discussion in section 3.3), one expects the other peaks at radii well outside the beam core to lie at the radii $r_C/(2n+1)$, for integers n (since the peak corresponds to the particle making a quarter oscillation plus any multiple of a half oscillation). One further expects the height of the peaks to be governed by equation (12) for large radii, and thus their heights are given by

$$\Delta E_{\max} \frac{\ln \dfrac{r_C}{c_0 \sigma_\perp} - \ln(2n+1)}{\ln \dfrac{r_C}{c_0 \sigma_\perp}}.$$

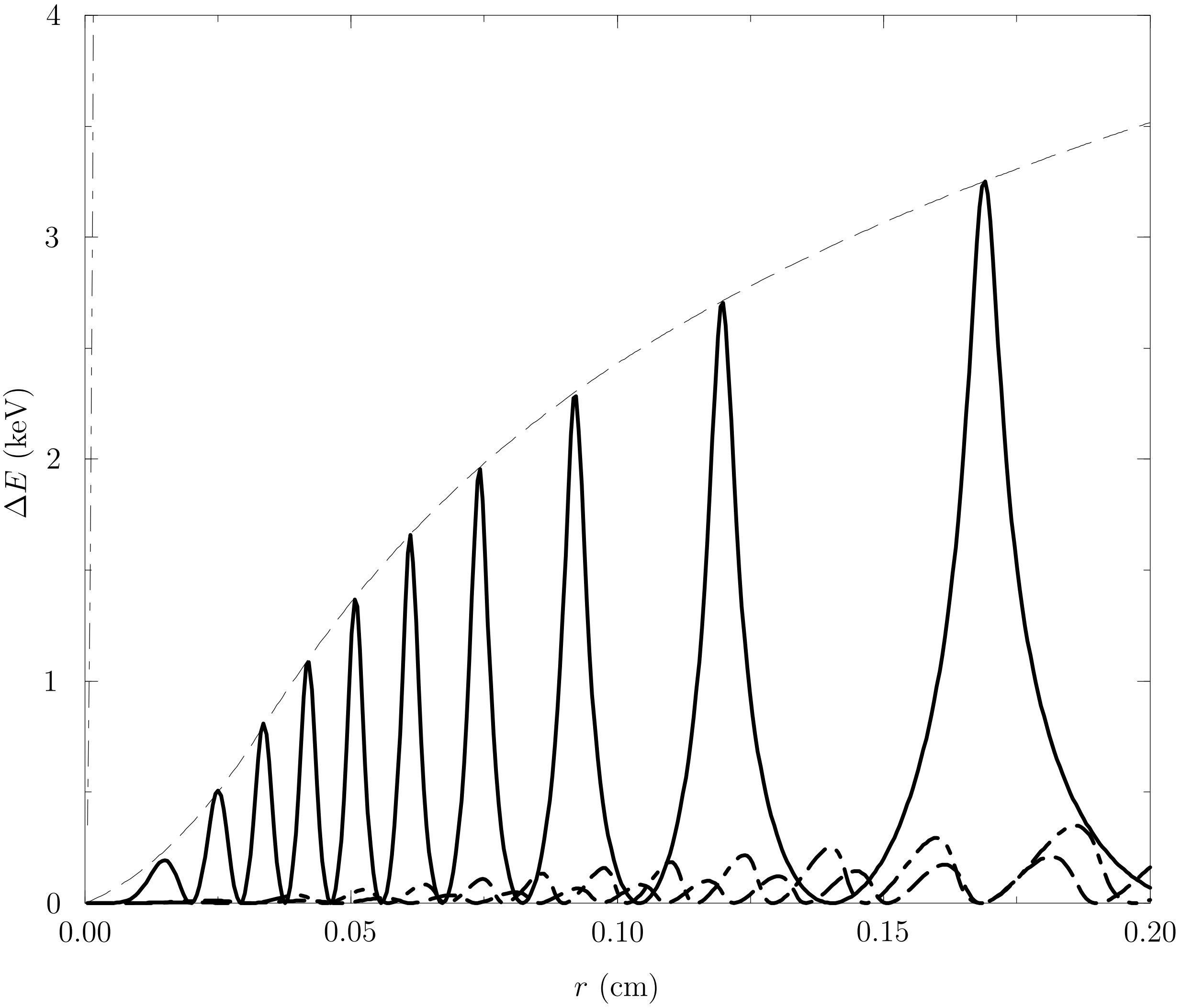

Figure 3: Same as figure 1, showing detail at low radii.

Examining the figures shows there is excellent agreement between the expectations given here for the expected radii and height of the peaks at radii well outside the beam core and the numerical results. Note that there are a finite number of peaks in the figures, which is $\lfloor 2n_O(0) \rfloor = \lfloor 2/[\lambda_{\max}\Delta t_{\text{step}}] \rfloor$ (see section 4.1 and figure 3).

One of the most dramatic effects seen in the figures is the result of computing the behavior for a non-rectangular bunch. At large radii, the shape of the bunch ceases to matter, as shown in equation (3), and confirmed in figures 1 and 2. However, if either the Gaussian or the parabolic-like longitudinal bunch shape is used, the energy gain is greatly reduced for lower radii from what it would be for the rectangular bunch (also see figure 3). This is because the current in the bunch is often less than the peak current, which is used throughout for the rectangular bunch shape. For the very low radii where the test charge goes through several oscillations, there is probably a sort of adiabatic invariance (see, for example, [5]) of the action due to the fact that the oscillation period is short compared to the time scale over which the bunch current varies, which causes the particle to gain very little energy (see figure 3).

Despite the large difference seen between the curves for rectangular and non-rectangular bunches, the total energy deposited doesn't change so much. There are several reasons behind this. The first is that for a uniform distribution of test particles, there are more particles at large radii. For large radii, there is little difference between the energy gained for the different longitudinal distributions. The second reason is that the peaks for the rectangular distribution tend to be narrower than the peaks for the non-rectangular distributions, thus not giving as much of a change in area under the peak as one might expect. Finally, the peaks for the rectangular distribution are rather widely spaced for the largest ones.

Thus, the main effect of using a realistic longitudinal distribution instead of a rectangular one is to reduce the maximum energy the test charges will gain. Additionally, the effect on the average energy deposition could be much stronger if there were a nonuniform distribution of particles in the beam pipe, particularly if that distribution were concentrated more at lower radii.

Finally, note that the program which produced the results in this section can be obtained from `http://wwwslap/collective/jsberg/egain/`.

5 Conclusions

This paper has given a derivation of an approximate formula for the energy gained by a particle when a bunch passes by. Equations (3), (8), and (12) describe the energy gained, and the transition radius (11) indicates which approximation applies.

When the results are numerically integrated, one finds a significant dependence of the energy gain on the longitudinal bunch distribution. In particular, the peak values of the energy gain will be significantly reduced if one uses a realistic distribution instead of a rectangular one.

If before the bunch passes by, there is a uniform distribution of test charges in the beam pipe, the sum of (13) and (14) gives the total energy gained by the test charges in most cases. If the transition radius is within the beam pipe, the energy deposition per test charge is proportional to the square of the beam current. Taking into account the fact that when the test charges in question are photoelectrons produced by synchrotron radiation, the number of source particles N_T will also be proportional to the beam current, the total energy deposition will be proportional to the cube of the beam current. In practice, the total energy deposition is not reduced greatly when one uses a realistic distribution instead of a rectangular one, but that might change if the initial distribution were not uniform, but concentrated more at lower radii.

If instead one is interested in the energy gained by the photoelectrons which are initially produced as the first bunch goes by, that can be computed reliably (for any longitudinal distribution) using equation (6), and the fact that the energies of the particles will be distributed from 0 to this value according to the distribution function $1/(2\sqrt{E\Delta E_{\text{wall}}})$ (see section 3.1 and appendix A).

Finally, note that when integrating the equations of motion, the step sizes described in section 4.1 must be used, especially for particles starting at low radii, otherwise the results could be very inaccurate.

There are many improvements which could be made in this analysis. One could attempt to perform an adiabatic invariance analysis to attempt to analytically approximate what the energy gains are for a non-rectangular distribution. One could consider what happens if particles have an initial radial momentum or angular momentum. One could consider what happens in the presence of a non-cylindrically symmetric beam (which will almost certainly be the case due to non-equal horizontal and vertical beta functions). One could look at what happens in the presence of a magnetic field, as there will be in the dipoles and quadrupoles. One could compute the longitudinal momentum that the particles gain due to the bunch passage. So, there is still much work to be done.

5.1 Acknowledgments

The author would like to thank Francesco Ruggiero for many helpful suggestions and his thorough proofreading and checking of the results, and Oliver Brüning for a helpful discussion which pointed out an error in an earlier version of these results.

A The Probability Distribution for Photoelectron Energy Gain

As described in subsection 3.1, the momentum gained as a function of scaled longitudinal position $\bar{z}$ is

$$P = \Delta p_{\mathrm{wall}} \int_{-\infty}^{\bar{z}} \lambda(z')dz'. \tag{7}$$

If $\lambda(\bar{z})d\bar{z}$ is the probability that there is a source particle between the scaled longitudinal positions $\bar{z}$ and $\bar{z} + d\bar{z}$, the probability of a photoelectron being produced between $\bar{z}$ and $\bar{z} + d\bar{z}$ is also $\lambda(\bar{z})d\bar{z}$. If $f(P)dP$ is the probability that the momentum gain is between P and $P + dP$, $\lambda(\bar{z})$ and $f(P)$ must be related by $\lambda(\bar{z})d\bar{z} = f(P)dP$ along with the relation (7). Thus, $f(P) = \lambda(\bar{z})/(dP/d\bar{z})$. But from equation (7), $dP/d\bar{z} = \Delta p_{\mathrm{wall}}\lambda(\bar{z})$. Thus, $f(P) = 1/\Delta p_{\mathrm{wall}}$. Clearly, P can potentially be anywhere from 0 to Δp_{wall}. Thus, the momentum is uniformly distributed from 0 to Δp_{wall}.

Next, following a similar argument, since the energy gain $E = P^2/2m_T$, if $g(E)$ is the distribution in energy,

$$g(E) = \frac{f(P)}{\frac{dE}{dP}} = \frac{m_T f(P)}{P} = \frac{m_T}{P\Delta p_{\mathrm{wall}}} = \frac{1}{2\sqrt{E\Delta E_{\mathrm{wall}}}}.$$

This is nonzero only between 0 and ΔE_{wall}. The average of E is $\Delta E_{\mathrm{wall}}/3$.

Note that this result was found previously by S. A. Heifets in [2] (see equation (13) there and the surrounding paragraph).

References

[1] Ohmi, K., "Beam-Photoelectron Interactions in Positron Storage Rings," Phys. Rev. Lett. **75**, 1526–9 (1995).

[2] Heifets, S. A., "Study of an Instability of the PEP-II Positron Beam (Ohmi Effect and Multipactoring)," in "Proceedings of the International Workshop on Collective Effects and Impedance for B-Factories (CEIBA95)," KEK Proceedings 96-6, KEK, Tsukuba, Japan (1995), 295–321.

[3] Zimmermann, Frank, "A Simulation Study of Electron-Cloud Instability and Beam-Induced Multipacting in the LHC," LHC Project Report 95, CERN, Geneva, Switzerland (1997).

[4] Gröbner, O., "Beam Induced Multipacting," to appear in the Proceedings of the 1997 Particle Accelerator Conference (1997).

[5] Lichtenberg, A. J. and M. A. Lieberman, "Regular and Stochastic Motion," Springer-Verlag, New York (1983).

[6] Forest, Étienne and Ronald D. Ruth, "Fourth-Order Symplectic Integration," Physica D **43** (1990), 105–117. Suzuki, Masuo, "Fractal Decomposition of Exponential Operators with Applications to Many-Body Theories and Monte Carlo Simulations," Phys. Lett. A **146** (1990), 319–323. Yoshida, Haruo, "Construction of Higher Order Symplectic Integrators," Phys. Lett. A **150** (1990), 262–268.