Coherent static tune shift due to electron cloud (Draft 3)

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1 General tune shift formula

In lowest order approximation, the coherent tune shift due to an electric field perturbation $E_{x(y)}$ acting on the beam is given by

$$\Delta Q_{x(y)} = \frac{e}{4\pi E} \oint ds\ \beta_{x(y)} \frac{\partial \bar{E}_{x(y)}}{\partial \Delta x(y)},$$

in which E is the beam energy, β is the lattice function, and $\bar{E}_{x(y)}(\Delta x, \Delta y)$ is the value of the electric field produced by a cloud charge density centered at x_c, y_c, acting on a beam centered at x_b, y_b, averaged over the beam distibution. Here $\Delta x = x_b - x_c$, and $\Delta y = y_b - y_c$. The tricky bit is calculating $\bar{E}_{x(y)}(\Delta x, \Delta y)$. The calculation is outlined below.

2 Electric field

The potential of a general three-dimensional charge distribution $\rho(x, y, z)$

$$\phi(x, y, z) = \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx'\ dy'\ dz'\ \frac{\rho(x', y', z')}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}$$

Using

$$\frac{1}{s} = \frac{2}{\sqrt{\pi}} \int_0^{\infty} du\ \mathrm{e}^{-s^2 u^2},$$

this becomes

$$\phi(x, y, z) = \frac{1}{4\pi\varepsilon_0} \frac{2}{\sqrt{\pi}} \int_0^{\infty} du\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx'\ dy'\ dz'\ \rho(x', y', z') \mathrm{e}^{-u^2\left((x-x')^2+(y-y')^2+(z-z')^2\right)}.$$

For a cloud charge distribution which is uniform in z, but otherwise general, and centered at x_c and y_c, then

$$\rho(x, y, z) = \lambda g(x - x_c, y - y_c),$$

in which λ is the charge per unit length in the z direction, and $g(x, y)$ describes the transverse shape of the density. Then the potential is

$$\phi(x, y, z) = \frac{\lambda}{4\pi\varepsilon_0} \frac{2}{\sqrt{\pi}} \int_0^{\infty} du\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx'\ dy'\ dz'\ g(x' - x_c, y' - y_c) \mathrm{e}^{-u^2\left((x-x')^2+(y-y')^2+(z-z')^2\right)}.$$

We now make a change of integration variables to

$$q = \frac{1}{u^2},\ u = \frac{1}{\sqrt{q}},\ du = -\frac{dq}{2q^{3/2}}.$$

$$\phi(x,y,z) = \frac{-\lambda}{4\pi\varepsilon_0}\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{dq}{q^{3/2}}\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'\ dy'\ dz'\ g(x'-x_c, y'-y_c)\mathrm{e}^{-\left((x-x')^2+(y-y')^2+(z-z')^2\right)/q}.$$

Using

$$\frac{1}{\sqrt{\pi}\sqrt{q}}\int_{-\infty}^{\infty} dz'\ \mathrm{e}^{-(z-z')^2/q} = 1,$$

$$\phi(x,y) = \frac{-\lambda}{4\pi\varepsilon_0}\int_0^\infty \frac{dq}{q}\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'\ dy' g(x'-x_c, y'-y_c)\mathrm{e}^{-\left((x-x')^2+(y-y')^2\right)/q}.$$

With the replacement $x' \to x' + x_c$, $y' \to y' + y_c$, then

$$\phi(x,y) = \frac{-\lambda}{4\pi\varepsilon_0}\int_0^\infty \frac{dq}{q}\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'\ dy' g(x', y')\mathrm{e}^{-\left((x-x_c-x')^2+(y-y_c-y')^2\right)/q}.$$

2.1 Electric field in the x-direction

The electric field in the x-direction is

$$E_x = -\frac{\partial\phi}{\partial x} = \frac{-\lambda}{2\pi\varepsilon_0}\int_0^\infty \frac{dq}{q^2}\ \int_{-\infty}^{\infty} dx'\ (x-x_c-x')\int_{-\infty}^{\infty} dy' g(x', y')\mathrm{e}^{-\left((x-x_c-x')^2+(y-y_c-y')^2\right)/q}.$$

Let the beam distribution be given transversely by a Gaussian, centered at x_b and y_b, with rms sizes σ_x and σ_y:

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}\mathrm{e}^{-(x-x_b)^2/(2\sigma_x^2)-(y-y_b)^2/(2\sigma_y^2)}$$

This distribution is normalized to unity. Then the average value of the field is

$$\bar{E}_x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\ dy\ f(x,y)E_x(x,y)$$

Explicitly,

$$\begin{aligned}\bar{E}_x \quad = \quad & \frac{1}{2\pi\sigma_x\sigma_y}\frac{-\lambda}{2\pi\varepsilon_0}\int_0^\infty \frac{dq}{q^2}\int_{-\infty}^{\infty} dx\int_{-\infty}^{\infty} dy\ \int_{-\infty}^{\infty} dx'\ (x-x_c-x')\int_{-\infty}^{\infty} dy' g(x', y')\ \times \\ & \mathrm{e}^{-\left((x-x_c-x')^2+(y-y_c-y')^2\right)/q-(x-x_b^2)/(2\sigma_x^2)-(y-y_b)^2/(2\sigma_y^2)}\end{aligned}$$

We can do the x and y integrations using

$$\frac{1}{\sqrt{2\pi q}\sigma_x}\int_{-\infty}^{\infty} dx(x-x_c-x')\mathrm{e}^{-\left((x-x_c-x')^2\right)/q-(x-x_b^2)/(2\sigma_x^2)} = e^{-(x_c+x'-x_b)^2/(2\sigma_x^2+q)}\frac{(x_b-x_c-x')q}{(2\sigma_x^2+q)^{3/2}}$$

$$\frac{1}{\sqrt{2\pi q}\sigma_y}\int_{-\infty}^{\infty} dy\mathrm{e}^{-\left((y-y_c-y')^2\right)/q-(y-y_b^2)/(2\sigma_y^2)} = e^{-(y_c+y'-y_b)^2/(2\sigma_y^2+q)}\frac{1}{(2\sigma_y^2+q)^{1/2}}$$

Thus, letting $\Delta x = x_b - x_c$ and $\Delta y = y_b - y_c$, we have

$$\bar{E}_x = \frac{-\lambda}{2\pi\varepsilon_0}\int_0^\infty dq\ \int_{-\infty}^\infty dx'\ \int_{-\infty}^\infty dy' g(x',y')\mathrm{e}^{-(\Delta x - x')^2/(2\sigma_x^2+q)-(\Delta y - y')^2/(2\sigma_y^2+q)}\frac{(\Delta x - x')}{(2\sigma_x^2+q)^{3/2}(2\sigma_y^2+q)^{1/2}}$$

The linear part of the field is

$$\begin{aligned}\left.\frac{\partial \bar{E}_x}{\partial \Delta x}\right|_{\Delta x=0,\Delta y=0} &= \frac{-\lambda}{2\pi\varepsilon_0}\int_0^\infty dq\ \int_{-\infty}^\infty dx'\ \int_{-\infty}^\infty dy' g(x',y')\mathrm{e}^{-x'^2/(2\sigma_x^2+q)-y'^2/(2\sigma_y^2+q)}\times \\ &\left(\frac{1}{(2\sigma_x^2+q)^{3/2}(2\sigma_y^2+q)^{1/2}} - \frac{2x'^2}{(2\sigma_x^2+q)^{5/2}(2\sigma_y^2+q)^{1/2}}\right) \\ &= \frac{-\lambda}{2\pi\varepsilon_0}\int_0^\infty dq\ \int_{-\infty}^\infty dx'\ \int_{-\infty}^\infty dy' g(x',y')\frac{\mathrm{e}^{-x'^2/(2\sigma_x^2+q)-y'^2/(2\sigma_y^2+q)}\left(2\sigma_x^2+q-2x'^2\right)}{(2\sigma_x^2+q)^{5/2}(2\sigma_y^2+q)^{1/2}}\end{aligned}$$

Thus,

$$\left.\frac{\partial \bar{E}_x}{\partial \Delta x}\right|_{\Delta x=0,\Delta y=0} = \frac{-\lambda}{2\pi\varepsilon_0}\int_{-\infty}^\infty dx'\ \int_{-\infty}^\infty dy' g(x',y')w_x(x',y'),$$

in which the weight function is

$$w_x(x,y) = \int_0^\infty dq\ \frac{\mathrm{e}^{-x^2/(2\sigma_x^2+q)-y^2/(2\sigma_y^2+q)}\left(2\sigma_x^2+q-2x^2\right)}{(2\sigma_x^2+q)^{5/2}(2\sigma_y^2+q)^{1/2}}$$

If we define

$$p = \frac{q}{2\sigma_x^2},$$

and

$$r = \frac{\sigma_y^2}{\sigma_x^2},$$

$$u = \frac{x}{\sqrt{2}\sigma_x},$$

$$v = \frac{y}{\sqrt{2}\sigma_y},$$

then

$$w_x(u,v) = \frac{1}{2\sigma_x^2}\int_0^\infty dp\ \frac{\mathrm{e}^{-u^2/(1+p)-v^2/(1+p/r)}\left(1+p-2u^2\right)}{(1+p)^{5/2}(r+p)^{1/2}} = \frac{\tilde{w}_x(u,v)}{2\sigma_x^2}$$

The linear part of the field is

$$\left.\frac{\partial \bar{E}_x}{\partial \Delta x}\right|_{\Delta x=0,\Delta y=0} = \frac{-\lambda\sigma_y}{2\pi\varepsilon_0\sigma_x}\int_{-\infty}^\infty du'\ \int_{-\infty}^\infty dv' g(\sqrt{2}\sigma_x u', \sqrt{2}\sigma_y v')\tilde{w}_x(u',v'),$$

In terms of the cloud 3D charge density $\rho(x,y) = \lambda g(x,y)$,

$$\left.\frac{\partial \bar{E}_x}{\partial \Delta x}\right|_{\Delta x=0,\Delta y=0} = \frac{-\sigma_y}{2\pi\varepsilon_0\sigma_x}\int_{-\infty}^\infty du'\ \int_{-\infty}^\infty dv' \rho(\sqrt{2}\sigma_x u', \sqrt{2}\sigma_y v')\tilde{w}_x(u',v').$$

2.1.1 Example

For example, suppose that the cloud distribution is a Gaussian:

$$g(x,y) = \frac{1}{2\pi ab} \mathrm{e}^{-x^2/2a^2 - y^2/2b^2}$$

Then

$$\left.\frac{\partial \bar{E}_x}{\partial \Delta x}\right|_{\Delta x=0, \Delta y=0} = \frac{-\lambda}{2\pi\varepsilon_0} \frac{\sigma_y}{2\pi ab\sigma_x} \int_{-\infty}^{\infty} du' \int_{-\infty}^{\infty} dv' \mathrm{e}^{-u'^2\sigma_x^2/a^2 - v'^2\sigma_y^2/b^2} \tilde{w}(u',v'),$$

The u' integration gives

$$\frac{1}{\sqrt{\pi}a} \int_{-\infty}^{\infty} du' \ \mathrm{e}^{-u'^2\sigma_x^2/a^2 - u'^2/(1+p)} \left(1+p-2u'^2\right) = \frac{(1+p)^{5/2}\sigma_x^2}{\left((1+p)\sigma_x^2+a^2\right)^{3/2}}$$

The v' integration gives

$$\frac{1}{\sqrt{\pi}b} \int_{-\infty}^{\infty} dv' \ \mathrm{e}^{-v'^2\sigma_y^2/b^2 - v'^2/(1+p/r)} = \frac{(1+p/r)^{1/2}}{\left((r+p)\sigma_x^2+b^2\right)^{1/2}}$$

so

$$\left.\frac{\partial \bar{E}_x}{\partial \Delta x}\right|_{\Delta x=0, \Delta y=0} = \frac{-\lambda\sigma_x\sigma_y}{4\pi\varepsilon_0} \int_0^{\infty} dp \ \frac{1}{\left((1+p)\sigma_x^2+a^2\right)^{3/2} \left((r+p)\sigma_x^2+b^2\right)^{1/2}} = \frac{-\lambda}{2\pi\varepsilon_0 \left(\sigma_x^2+a^2+\sqrt{(\sigma_x^2+a^2)(\sigma_y^2+b^2)}\right)}$$

2.2 Electric field in the y-direction

The electric field in the y-direction is

$$E_y = -\frac{\partial\phi}{\partial y} = \frac{-\lambda}{2\pi\varepsilon_0} \int_0^{\infty} \frac{dq}{q^2} \int_{-\infty}^{\infty} dx' \ (y-y_c-y') \int_{-\infty}^{\infty} dy' g(x',y') \mathrm{e}^{-\left((x-x_c-x')^2+(y-y_c-y')^2\right)/q}.$$

With the same assumption of a Gaussian beam, the average value of the field is

$$\begin{aligned}\bar{E}_y \ &= \ \frac{1}{2\pi\sigma_x\sigma_y} \frac{-\lambda}{2\pi\varepsilon_0} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_0^{\infty} \frac{dq}{q^2} \int_{-\infty}^{\infty} dy' \ (y-y_c-y') \int_{-\infty}^{\infty} dx' g(x',y') \times \\ &\quad \mathrm{e}^{-\left((x-x_c-x')^2+(y-y_c-y')^2\right)/q - (x-x_b^2)/(2\sigma_x^2) - (y-y_b)^2/(2\sigma_y^2)}\end{aligned}$$

We can do the x and y integrations using

$$\frac{1}{\sqrt{2\pi q}\sigma_y} \int_{-\infty}^{\infty} dy(y-y_c-y') \mathrm{e}^{-\left((y-y_c-y')^2\right)/q - (y-y_b^2)/(2\sigma_y^2)} = e^{-(y_c+y'-y_b)^2/(2\sigma_y^2+q)} \frac{(y_b-y_c-y')q}{(2\sigma_y^2+q)^{3/2}}$$

$$\frac{1}{\sqrt{2\pi q}\sigma_y} \int_{-\infty}^{\infty} dx \mathrm{e}^{-\left((x-x_c-x')^2\right)/q - (x-x_b^2)/(2\sigma_x^2)} = e^{-(x_c+x'-x_b)^2/(2\sigma_x^2+q)} \frac{1}{(2\sigma_x^2+q)^{1/2}}$$

Thus, letting $\Delta x = x_b - x_c$ and $\Delta y = y_b - y_c$, we have

$$\bar{E}_y = \frac{-\lambda}{2\pi\varepsilon_0} \int_0^{\infty} dq \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' g(x',y') \mathrm{e}^{-(\Delta x-x')^2/(2\sigma_x^2+q) - (\Delta y-y')^2/(2\sigma_y^2+q)} \frac{(\Delta y-y')}{(2\sigma_y^2+q)^{3/2}(2\sigma_x^2+q)^{1/2}}$$

The linear part of this is

$$
\begin{aligned}
\left.\frac{\partial \bar{E}_y}{\partial \Delta y}\right|_{\Delta x=0,\Delta y=0} &= \frac{-\lambda}{2\pi\varepsilon_0}\int_0^\infty dq\ \int_{-\infty}^{\infty} dx'\ \int_{-\infty}^{\infty} dy' g(x',y')\mathrm{e}^{-x'^2/(2\sigma_x^2+q)-y'^2/(2\sigma_y^2+q)} \times \\
&\quad \left(\frac{1}{(2\sigma_y^2+q)^{3/2}(2\sigma_x^2+q)^{1/2}} - \frac{2y'^2}{(2\sigma_y^2+q)^{5/2}(2\sigma_x^2+q)^{1/2}}\right) \\
&= \frac{\lambda}{2\pi\varepsilon_0}\int_0^\infty dq\ \int_{-\infty}^{\infty} dx'\ \int_{-\infty}^{\infty} dy' g(x',y')\frac{\mathrm{e}^{-x'^2/(2\sigma_x^2+q)-y'^2/(2\sigma_y^2+q)}\left(2\sigma_y^2+q-2y'^2\right)}{(2\sigma_y^2+q)^{5/2}(2\sigma_x^2+q)^{1/2}}
\end{aligned}
$$

Thus,

$$
\left.\frac{\partial \bar{E}_y}{\partial \Delta y}\right|_{\Delta x=0,\Delta y=0} = \frac{-\lambda}{2\pi\varepsilon_0}\int_{-\infty}^{\infty} dx'\ \int_{-\infty}^{\infty} dy' g(x',y')w_y(x',y'),
$$

in which the weight function is

$$
w_y(x,y) = \int_0^\infty dq\ \frac{\mathrm{e}^{-x^2/(2\sigma_x^2+q)-y^2/(2\sigma_y^2+q)}\left(2\sigma_y^2+q-2y^2\right)}{(2\sigma_y^2+q)^{5/2}(2\sigma_x^2+q)^{1/2}}
$$

If we define

$$
p = \frac{q}{2\sigma_x^2},
$$

and

$$
\begin{aligned}
r &= \frac{\sigma_y^2}{\sigma_x^2}, \\
u &= \frac{x}{\sqrt{2}\sigma_x}, \\
v &= \frac{y}{\sqrt{2}\sigma_y},
\end{aligned}
$$

then

$$
w_y(u,v) = \frac{1}{2\sigma_x^2}\int_0^\infty dp\ \frac{\mathrm{e}^{-u^2/(1+p)-v^2/(1+p/r)}\left(r+p-2rv^2\right)}{(r+p)^{5/2}(1+p)^{1/2}} = \frac{\tilde{w}_y(u,v)}{2\sigma_x^2}
$$

The linear part of the field is

$$
\left.\frac{\partial \bar{E}_y}{\partial \Delta y}\right|_{\Delta x=0,\Delta y=0} = \frac{-\lambda\sigma_y}{2\pi\varepsilon_0\sigma_x}\int_{-\infty}^{\infty} du'\ \int_{-\infty}^{\infty} dv' g(\sqrt{2}\sigma_x u', \sqrt{2}\sigma_y v')\tilde{w}_y(u',v').
$$

2.2.1 Example

For example, suppose that the cloud distribution is a Gaussian:

$$
g(x,y) = \frac{1}{2\pi ab}\mathrm{e}^{-x^2/2a^2-y^2/2b^2}
$$

Then

$$
\left.\frac{\partial \bar{E}_y}{\partial \Delta y}\right|_{\Delta x=0,\Delta y=0} = \frac{-\lambda\sigma_y}{2\pi\varepsilon_0\sigma_x}\frac{1}{2\pi ab}\int_{-\infty}^{\infty} du'\ \int_{-\infty}^{\infty} dv' \mathrm{e}^{-u'^2\sigma_x^2/a^2-v'^2\sigma_y^2/b^2}\tilde{w}_y(u',v'),
$$

The u' integration gives

$$\frac{1}{\sqrt{\pi}a}\int_{-\infty}^{\infty} du'\ \mathrm{e}^{-u'^2\sigma_x^2/a^2-u'^2/(1+p)} = \frac{(1+p)^{1/2}}{\left((1+p)\sigma_x^2+a^2\right)^{1/2}}$$

The v' integration gives

$$\frac{1}{\sqrt{\pi}b}\int_{-\infty}^{\infty} dv'\ \mathrm{e}^{-v'^2\sigma_y^2/b^2-v'^2/(1+p/r)}\left(r+p-2rv'^2)\right) = \frac{\sigma_x^2(p+r)^{5/2}}{\sqrt{r}\left((r+p)\sigma_x^2+b^2\right)^{3/2}}$$

so

$$\left.\frac{\partial \bar{E}_y}{\partial \Delta y}\right|_{\Delta x=0,\Delta y=0} = \frac{-\lambda\sigma_x^2}{4\pi\varepsilon_0}\int_0^{\infty} dp\ \frac{1}{\left((1+p)\sigma_x^2+a^2\right)^{1/2}\left((r+p)\sigma_x^2+b^2\right)^{3/2}} = \frac{-\lambda}{2\pi\varepsilon_0\left(\sigma_y^2+b^2+\sqrt{(\sigma_x^2+a^2)(\sigma_y^2+b^2)}\right)}$$

3 Summary

The coherent tune shift is

$$\Delta Q_{x(y)} = \frac{e}{4\pi E}\oint ds\ \beta_{x(y)}\frac{\partial \bar{E}_{x(y)}}{\partial \Delta x(y)},$$

in which $E = m_0c^2\gamma$ is the beam energy. The beam is assumed to be a Gaussian with rms sizes σ_x, σ_y. In the x direction, the linear gradient of the electric field due to the cloud, averaged over the beam distribution, is

$$\frac{\partial \bar{E}_x}{\partial \Delta x} = \frac{-\sigma_y}{2\pi\varepsilon_0\sigma_x}\int_{-\infty}^{\infty} du' \int_{-\infty}^{\infty} dv' \rho(\sqrt{2}\sigma_x u', \sqrt{2}\sigma_y v')\tilde{w}_x(u', v'),$$

in which $\rho(x, y)$ is the 3-dimensional cloud charge density (assumed to be uniform in the direction of the beam), and the weight function is

$$\tilde{w}_x(u, v) = \int_0^{\infty} dp\ \frac{\mathrm{e}^{-u^2/(1+p)-v^2/(1+p/r)}\left(1+p-2u^2\right)}{(1+p)^{5/2}(r+p)^{1/2}},$$

and $r = \sigma_y^2/\sigma_x^2$. In the y direction,

$$\frac{\partial \bar{E}_y}{\partial \Delta y} = \frac{-\sigma_y}{2\pi\varepsilon_0\sigma_x}\int_{-\infty}^{\infty} du' \int_{-\infty}^{\infty} dv' \rho(\sqrt{2}\sigma_x u', \sqrt{2}\sigma_y v')\tilde{w}_y(u', v'),$$

$$\tilde{w}_y(u, v) = \int_0^{\infty} dp\ \frac{\mathrm{e}^{-u^2/(1+p)-v^2/(1+p/r)}\left(r+p-2ru^2\right)}{(r+p)^{5/2}(1+p)^{1/2}}.$$

Note that the above approach can also be used to estimate the incoherent tune spread, which can be relevant for emittance growth.

3.1 Gaussian cloud example

For the example of a Gaussian cloud density given above, we have for the horizontal tune shift

$$\Delta Q_x = \frac{-e}{8\pi^2\varepsilon_0 E}\oint ds \; \frac{\beta_x\lambda}{\left(\sigma_x^2+a^2+\sqrt{(\sigma_x^2+a^2)(\sigma_y^2+b^2)}\right)}$$

In terms of the peak 3D cloud number density,

$$\rho_{n,max} = \frac{-\lambda}{2\pi e a b},$$

$$\Delta Q_x = \frac{e^2}{4\pi\varepsilon_0 E}\oint ds \; \frac{\beta_x\rho_{n,max}ab}{\left(\sigma_x^2+a^2+\sqrt{(\sigma_x^2+a^2)(\sigma_y^2+b^2)}\right)}$$

Using

$$r_e = \frac{e^2}{4\pi\varepsilon_0 m_0 c^2},$$

this can be written

$$\Delta Q_x = \frac{r_e}{\gamma}\oint ds \; \frac{\beta_x\rho_{n,max}ab}{\left(\sigma_x^2+a^2+\sqrt{(\sigma_x^2+a^2)(\sigma_y^2+b^2)}\right)}.$$

Similarly,

$$\Delta Q_y = \frac{r_e}{\gamma}\oint ds \; \frac{\beta_y\rho_{n,max}ab}{\left(\sigma_y^2+b^2+\sqrt{(\sigma_x^2+a^2)(\sigma_y^2+b^2)}\right)}.$$

If the size of the cloud is much bigger than the beam in both x and y, then we have

$$\Delta Q_x \approx \frac{r_e}{\gamma}\oint ds \; \frac{\beta_x\rho_{n,max}ab}{(a^2+ab)} = \frac{r_e}{\gamma}\oint ds \; \frac{\beta_x\rho_{n,max}}{(1+a/b)},$$

and

$$\Delta Q_y \approx \frac{r_e}{\gamma}\oint ds \; \frac{\beta_y\rho_{n,max}ab}{(b^2+ab)} = \frac{r_e}{\gamma}\oint ds \; \frac{\beta_y\rho_{n,max}}{(1+b/a)}.$$

3.2 Line charge example

Suppose that the cloud charge density is a line charge located at $x = a$. Then we have

$$\rho(x,y) = \lambda\delta(x-a)\delta(y)$$

and the average electric field gradient in the x-direction is

$$\frac{\partial\bar{E}_x}{\partial\Delta x} = \frac{-\sigma_y\lambda}{2\pi\varepsilon_0\sigma_x}\int_{-\infty}^{\infty} du' \int_{-\infty}^{\infty} dv'\delta(\sqrt{2}\sigma_x u'-a)\delta(\sqrt{2}\sigma_y v')\tilde{w}_x(u',v').$$

$$\frac{\partial\bar{E}_x}{\partial\Delta x} = \frac{-\lambda}{4\pi\varepsilon_0\sigma_x^2}\tilde{w}_x(a/\sqrt{2}\sigma_x,0) = \frac{-\lambda}{4\pi\varepsilon_0\sigma_x^2}\int_0^{\infty} dp \; \frac{\mathrm{e}^{-a^2/2\sigma_x^2(1+p)}\left(1+p-a^2/\sigma_x^2\right)}{(1+p)^{5/2}(r+p)^{1/2}}.$$

For the case of a round beam, $r = 1$, we have

$$\frac{\partial \bar{E}_x}{\partial \Delta x} = \frac{-\lambda}{4\pi\varepsilon_0\sigma_x^2}\int_0^\infty dp\, \frac{\mathrm{e}^{-a^2/2\sigma_x^2(1+p)}\left(1+p-a^2/\sigma_x^2\right)}{(1+p)^3} = \frac{-\lambda}{2\pi\varepsilon_0}\left(-\frac{1}{a^2}+\mathrm{e}^{-a^2/2\sigma_x^2}(1/a^2+1/\sigma_x^2)\right).$$

If

$$a \gg \sigma_x,$$

then

$$\frac{\partial \bar{E}_x}{\partial \Delta x} = \frac{\lambda}{2\pi\varepsilon_0 a^2}.$$

This will be negative for an electron cloud.

The average electric field gradient in the y-direction is

$$\frac{\partial \bar{E}_y}{\partial \Delta y} = \frac{-\sigma_y\lambda}{2\pi\varepsilon_0\sigma_x}\int_{-\infty}^{\infty} du' \int_{-\infty}^{\infty} dv' \delta(\sqrt{2}\sigma_x u' - a)\delta(\sqrt{2}\sigma_y v')\tilde{w}_y(u',v').$$

$$\frac{\partial \bar{E}_y}{\partial \Delta y} = \frac{-\lambda}{4\pi\varepsilon_0\sigma_x^2}\tilde{w}_y(a/\sqrt{2}\sigma_x, 0) = \frac{-\lambda}{4\pi\varepsilon_0\sigma_x^2}\int_0^\infty dp\, \frac{\mathrm{e}^{-a^2/2\sigma_x^2(1+p)}}{(1+p)^{1/2}(r+p)^{3/2}}.$$

For the case of a round beam, $r = 1$, we have

$$\frac{\partial \bar{E}_y}{\partial \Delta y} = \frac{-\lambda}{4\pi\varepsilon_0\sigma_x^2}\int_0^\infty dp\, \frac{\mathrm{e}^{-a^2/2\sigma_x^2(1+p)}}{(1+p)^2} = \frac{-\lambda}{2\pi\varepsilon_0 a^2}\left(1-\mathrm{e}^{-a^2/2\sigma_x^2}\right).$$

If

$$a \gg \sigma_x,$$

then

$$\frac{\partial \bar{E}_y}{\partial \Delta y} = -\frac{\lambda}{2\pi\varepsilon_0 a^2}.$$

This will be positive for an electron cloud.

3.3 Uniform charge density example

Suppose that the cloud charge density is a uniform charge density ρ_0 over a circle of radius a, and the beam is round. Then the average electric field gradient in the x-direction is

$$\frac{\partial \bar{E}_x}{\partial \Delta x} = \frac{-\rho_0}{2\pi\varepsilon_0}\int_0^{2\pi} d\theta \int_0^{a/\sqrt{2}\sigma_x} r\, dr \tilde{w}_x(r\cos\theta, r\sin\theta).$$

$$\frac{\partial \bar{E}_x}{\partial \Delta x} = \frac{-\rho_0}{2\pi\varepsilon_0}\int_0^{2\pi} d\theta \int_0^{a/\sqrt{2}\sigma_x} r\, dr \int_0^\infty dp\, \frac{\mathrm{e}^{-r^2/(1+p)}\left(1+p-2r^2\cos\theta^2\right)}{(1+p)^3}$$

$$\frac{\partial \bar{E}_x}{\partial \Delta x} = \frac{-\rho_0}{2\varepsilon_0}\left(1-\mathrm{e}^{-a^2/2\sigma_x^2}\right).$$

If

$$a \gg \sigma_x,$$

then

$$\frac{\partial \bar{E}_x}{\partial \Delta x} = \frac{-\rho_0}{2\varepsilon_0}.$$

This will be positive for an electron cloud. This be compared with the case for a Gaussian cloud, for which

$$\left.\frac{\partial \bar{E}_x}{\partial \Delta x}\right|_{\Delta x=0,\Delta y=0} = \frac{-\lambda}{2\pi\varepsilon_0\,(2a^2)} = \frac{-\rho_{max}}{2\varepsilon_0}$$

$$\Delta Q_x \leqslant 0,\ \Delta Q_y \geqslant 0.$$

3.4 Round beam example

Suppose that the beam is round. This corresponds to $r = 1$, so the weight functions are

$$\tilde{w}_x(u,v) = \int_0^\infty dp\ \frac{\mathrm{e}^{-u^2/(1+p)-v^2/(1+p)}\left(1+p-2u^2\right)}{(1+p)^3},$$

$$\tilde{w}_y(u,v) = \int_0^\infty dp\ \frac{\mathrm{e}^{-u^2/(1+p)-v^2/(1+p)}\left(r+p-2v^2\right)}{(1+p)^3} = \tilde{w}_x(v,u).$$

The integral over p can be done analytically, giving

$$\tilde{w}_x(u,v) = \frac{\mathrm{e}^{-(u^2+v^2)}}{(u^2+v^2)^2}\left(u^2(1+2(u^2+v^2)) - v^2\right) + \frac{v^2-u^2}{(u^2+v^2)^2}.$$

In terms of polar coordinates $u = r'\cos\phi$ and $v = r'\sin\phi$

$$\tilde{w}_x(r',\phi) = \frac{\mathrm{e}^{-r'^2}}{r'^2}\left(r'^2 + (1+r'^2)\cos 2\phi\right) - \frac{\cos 2\phi}{r'^2}$$

and

$$\tilde{w}_y(r',\phi) = \frac{\mathrm{e}^{-r'^2}}{r'^2}\left(r'^2 - (1+r'^2)\cos 2\phi\right) + \frac{\cos 2\phi}{r'^2}$$

The field gradient is then

$$\left.\frac{\partial \bar{E}_x}{\partial \Delta x}\right|_{\Delta x=0,\Delta y=0} = \frac{-1}{2\pi\varepsilon_0}\int_0^{2\pi} d\phi\ \int_0^\infty r'dr'\rho(\sqrt{2}\sigma r'\cos\phi, \sqrt{2}\sigma r'\sin\phi)\tilde{w}_x(r',\phi).$$

This can also be written in terms of $r = r'/\sqrt{2}\sigma$ as

$$\left.\frac{\partial \bar{E}_x}{\partial \Delta x}\right|_{\Delta x=0,\Delta y=0} = \frac{-1}{2\pi\varepsilon_0}\int_0^{2\pi} d\phi\ \int_0^\infty r dr \rho(r\cos\phi, r\sin\phi)\frac{\tilde{w}_x(\frac{r}{\sqrt{2}\sigma},\phi)}{2\sigma^2}.$$

3.5 Line charge beam

If the beam size σ is much smaller than the other dimensions in the problem, we need to take the limit of the weight function when $\sigma \to 0$. The round beam weight function is

$$\begin{aligned}\frac{\tilde{w}_x(\frac{r}{\sqrt{2}\sigma},\phi)}{2\sigma^2} &= \frac{1}{2\sigma^2}\left(\frac{2\sigma^2 \mathrm{e}^{-\frac{r^2}{2\sigma^2}}}{r^2}\left(\frac{r^2}{2\sigma^2}+\left(1+\frac{r^2}{2\sigma^2}\right)\cos 2\phi\right)-\frac{2\sigma^2\cos 2\phi}{r^2}\right)\\ &= \frac{\mathrm{e}^{-\frac{r^2}{2\sigma^2}}}{2\sigma^2}\left(1+\left(\frac{2\sigma^2}{r^2}+1\right)\cos 2\phi\right)-\frac{\cos 2\phi}{r^2}\end{aligned}$$

Using

$$\lim_{\sigma\to 0}\frac{\mathrm{e}^{-\frac{r^2}{2\sigma^2}}}{2\sigma^2}=\frac{\delta(r)}{2r},$$

we have

$$w_{0,x}(r,\phi)=\lim_{\sigma\to 0}\frac{\tilde{w}_x(\frac{r}{\sqrt{2}\sigma},\phi)}{2\sigma^2}=\frac{\delta(r)}{2r}-\frac{\cos 2\phi}{r^2}.$$

The field gradient is then

$$\begin{aligned}\left.\frac{\partial \bar{E}_x}{\partial \Delta x}\right|_{\Delta x=0,\Delta y=0} &= \frac{-1}{2\pi\varepsilon_0}\int_0^{2\pi}d\phi\int_0^{\infty} r dr \rho(r\cos\phi, r\sin\phi) w_{0,x}(r,\phi)\\ &= \frac{-1}{2\pi\varepsilon_0}\left(\pi\rho(0,0)-\int_0^{2\pi}d\phi\cos 2\phi\int_0^{\infty}\frac{dr}{r}\rho(r\cos\phi, r\sin\phi)\right)\end{aligned}$$

Since

$$w_{0,y}(r,\phi)=\frac{\delta(r)}{2r}+\frac{\cos 2\phi}{r^2},$$

$$\left.\frac{\partial \bar{E}_y}{\partial \Delta y}\right|_{\Delta x=0,\Delta y=0} = \frac{-1}{2\pi\varepsilon_0}\left(\pi\rho(0,0)+\int_0^{2\pi}d\phi\cos 2\phi\int_0^{\infty}\frac{dr}{r}\rho(r\cos\phi, r\sin\phi)\right)$$

The difference of the field gradients, divided by the sum, is

$$\frac{1}{\rho(0,0)}\int_0^{2\pi}d\phi\cos 2\phi\int_0^{\infty}\frac{dr}{r}\rho(r\cos\phi, r\sin\phi)$$

$$\frac{\Delta Q_y-\Delta Q_x}{\Delta Q_y+\Delta Q_x}\approx\overline{\frac{\int_0^{2\pi}d\phi\cos 2\phi\int_0^{\infty}\frac{dr}{r}\rho(r\cos\phi, r\sin\phi)}{\rho(0,0)}}$$

In terms of Cartesian coordinates

$$\frac{\delta(r)}{r}=2\pi\delta(x)\delta(y),$$

and

$$\frac{\cos 2\phi}{r^2}=\frac{x^2-y^2}{(x^2+y^2)^2}$$

so

$$w_{0,x}(x,y) = \pi\delta(x)\delta(y) - \frac{x^2-y^2}{(x^2+y^2)^2}.$$

The field gradient is then

$$\begin{aligned}
\left.\frac{\partial \bar{E}_x}{\partial \Delta x}\right|_{\Delta x=0,\Delta y=0} &= \frac{-1}{2\pi\varepsilon_0}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \rho(x,y) w_{0,x}(x,y) \\
&= \frac{-1}{2\pi\varepsilon_0}\left(\pi\rho(0,0) - \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \rho(x,y)\frac{x^2-y^2}{(x^2+y^2)^2}.\right)
\end{aligned}$$

3.5.1 Gaussian cloud

Suppose that the cloud density is Gaussian:

$$\rho(x,y) = \frac{\lambda}{2\pi ab}\mathrm{e}^{-x^2/2a^2-y^2/2b^2}$$

Then the field gradient for a line charge beam is

$$\begin{aligned}
\left.\frac{\partial \bar{E}_x}{\partial \Delta x}\right|_{\Delta x=0,\Delta y=0} &= \frac{-\lambda}{2\pi\varepsilon_0}\left(\frac{1}{2ab} - \frac{1}{2\pi ab}\int_0^{2\pi} d\phi \cos 2\phi \int_0^{\infty}\frac{dr}{r}\mathrm{e}^{-r^2\left(\cos^2\phi/2a^2+\sin^2\phi/2b^2\right)}\right) \\
&= \frac{-\lambda}{2\pi\varepsilon_0}\left(\frac{1}{2ab} + \frac{1}{2ab} - \frac{1}{b(a+b)})\right) \\
&= \frac{-\lambda}{2\pi\varepsilon_0}\frac{1}{a(a+b)}
\end{aligned}$$

For the y field gradient, we have

$$\begin{aligned}
\left.\frac{\partial \bar{E}_y}{\partial \Delta y}\right|_{\Delta x=0,\Delta y=0} &= \frac{-\lambda}{2\pi\varepsilon_0}\left(\frac{1}{2ab} + \frac{1}{2\pi ab}\int_0^{2\pi} d\phi \cos 2\phi \int_0^{\infty}\frac{dr}{r}\mathrm{e}^{-r^2\left(\cos^2\phi/2a^2+\sin^2\phi/2b^2\right)}\right) \\
&= \frac{-\lambda}{2\pi\varepsilon_0}\left(\frac{1}{2ab} - \frac{1}{2ab} + \frac{1}{b(a+b)})\right) \\
&= \frac{-\lambda}{2\pi\varepsilon_0}\frac{1}{b(a+b)}
\end{aligned}$$

3.6 Electric field in the x-direction-line charge beam

The electric field in the x-direction is

$$E_x = -\frac{\partial\phi}{\partial x} = \frac{-\lambda}{2\pi\varepsilon_0}\int_0^{\infty}\frac{dq}{q^2}\int_{-\infty}^{\infty} dx' \ (x-x_c-x') \int_{-\infty}^{\infty} dy' g(x',y')\mathrm{e}^{-\left((x-x_c-x')^2+(y-y_c-y')^2\right)/q}.$$

Let the beam distribution be given transversely by a line charge, centered at x_b and y_b:

$$f(x,y) = \delta(x-x_b)\delta(y-y_b)$$

This distribution is normalized to unity. Then the average value of the field is

$$\bar{E}_x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\ dy\ f(x,y)E_x(x,y)$$

Explicitly, doing the integration over x and y,

$$\bar{E}_x \quad = \quad \frac{-\lambda}{2\pi\varepsilon_0}\int_0^{\infty}\frac{dq}{q^2}\ \int_{-\infty}^{\infty} dx'\ \ (x_b - x_c - x')\int_{-\infty}^{\infty} dy' g(x',y')\mathrm{e}^{-\left((x_b-x_c-x')^2+(y_b-y_c-y')^2\right)/q}$$

Letting $\Delta x = x_b - x_c$ and $\Delta y = y_b - y_c$, we have

$$\bar{E}_x = \frac{-\lambda}{2\pi\varepsilon_0}\int_0^{\infty}\frac{dq}{q^2}\ \int_{-\infty}^{\infty} dx'\ \int_{-\infty}^{\infty} dy' g(x',y')\mathrm{e}^{-(\Delta x-x')^2/q-(\Delta y-y')^2/q}(\Delta x - x')$$

The linear part of the field is

$$\left.\frac{\partial \bar{E}_x}{\partial \Delta x}\right|_{\Delta x=0,\Delta y=0} \quad = \quad \frac{-\lambda}{2\pi\varepsilon_0}\int_0^{\infty}\frac{dq}{q^3}\ \int_{-\infty}^{\infty} dx'\ \int_{-\infty}^{\infty} dy' g(x',y')(q-2x'^2)\mathrm{e}^{-(x'^2+y'^2)/q}$$

Thus,

$$\left.\frac{\partial \bar{E}_x}{\partial \Delta x}\right|_{\Delta x=0,\Delta y=0} = \frac{-\lambda}{2\pi\varepsilon_0}\int_{-\infty}^{\infty} dx'\ \int_{-\infty}^{\infty} dy' g(x',y')w_x(x',y'),$$

in which the weight function is

$$w_x(x,y) = \int_0^{\infty}\frac{dq}{q^3}(q-2x'^2)\mathrm{e}^{-(x'^2+y'^2)/q}$$

This integral diverges for $x = 0, y = 0$ and needs to be written in terms of a delta function-see above, previous section.

4 Sum rule

4.1 Derivation

From the Poisson equation, we have

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = \frac{e\rho(x,y)}{\varepsilon_0} = \frac{e\lambda g(x-x_c, y-y_c)}{\varepsilon_0}.$$

in which $\rho(x,y)$ is the cloud number density, and $g(x,y)$ describes the 2D shape of a cloud centered at x_c, y_c. Averaging over the beam distribution gives

$$\left\langle\frac{\partial E_x}{\partial x}\right\rangle + \left\langle\frac{\partial \bar{E}_y}{\partial y}\right\rangle = \frac{e\lambda}{\varepsilon_0}\frac{1}{2\pi\sigma_x\sigma_y}\int_{-\infty}^{\infty} dx\ \int_{-\infty}^{\infty} dy\ g(x-x_c, y-y_c)\mathrm{e}^{-(x-x_b)^2/(2\sigma_x^2)-(y-y_b)^2/(2\sigma_y^2)} = \frac{e\lambda}{\varepsilon_0}\langle g\rangle$$

If we make the approximation that

$$\beta_x \approx \beta_y \approx \beta,$$

then, since

$$\Delta Q_{x(y)} = \frac{e}{4\pi E} \oint ds\ \beta_{x(y)} \left\langle \frac{\partial E_{x(y)}}{\partial \Delta x(y)} \right\rangle,$$

we can write

$$\begin{aligned}
\Delta Q_x + \Delta Q_y &= \frac{e}{4\pi E} \oint ds\ \left(\beta_x \left\langle \frac{\partial E_x}{\partial x} \right\rangle + \beta_y \left\langle \frac{\partial E_y}{\partial y} \right\rangle \right) \\
&\approx \frac{e}{4\pi E} \oint ds\ \beta \left(\left\langle \frac{\partial E_x}{\partial x} \right\rangle + \left\langle \frac{\partial E_y}{\partial y} \right\rangle \right) \\
&\approx \frac{e^2 \lambda}{4\pi\varepsilon_0 E} \oint ds\ \beta \,\langle g \rangle = \frac{e^2}{4\pi\varepsilon_0 E} \oint ds\ \beta \,\langle \rho \rangle
\end{aligned}$$

Using

$$r_e = \frac{e^2}{4\pi\varepsilon_0 m_0 c^2},$$

this can be written

$$\Delta Q_x + \Delta Q_y \approx \frac{r_e}{\gamma} \oint ds\ \beta \,\langle \rho \rangle$$

This shows that the sum of the coherent tune shifts is approximately proportional to the integral around the ring of the cloud density, averaged over the beam, weighted with the local value of beta. If we use an overbar to indicate an averaging around the ring, as well as over the transverse beam dimensions, then we can write approximately that

$$\overline{\langle \rho \rangle} \approx \frac{\gamma(\Delta Q_x + \Delta Q_y)}{r_e \oint ds\ \beta}$$

4.2 Evaluation

To compare with a simulation, we wish to evaluate

$$\oint ds\ \beta \,\langle \rho \rangle = \overline{\langle \rho \rangle} \oint ds\ \beta$$

The ring is divided into n beamline elements, with index i. At each element, the radiation is characterized by the parameter I_i (photons per electron per meter, averaged over the element). The electron cloud density is a function of this parameter, and of the type of beamline element, designated by the index k (i.e., drift, dipole, wiggler, etc.): so at a given element

$$\langle \rho \rangle = f_k(I_i)$$

If there are m types of beamline elements, and n_k elements of each type, then the integral around the ring can thus be written as

$$\oint ds\ \beta \,\langle \rho \rangle = \sum_{k=1}^{m} \sum_{i=1}^{n_k} \beta_i L_i f_k(I_i).$$

If we make a simple linear approximation for the dependence of the density on the radiation parameter (an assumption which is likely to be wrong, given the nonlinear character of the electron cloud formation process), then we can write

$$f_k(I_i) = f_{0,k}\frac{I_i}{I_{0,k}}$$

in which $f_{0,k}$ is the density corresponding to the radiation parameter $I_{0,k}$, and

$$\oint ds\ \beta\left\langle\rho\right\rangle = \sum_{k=1}^{m}\frac{f_{0,k}}{I_{0,k}}\sum_{i=1}^{n_k}\beta_i L_i I_i$$

We can define the radiation parameter $I_{0,k}$ as being equal to the weighted average of I_i:

$$I_{0,k} = \frac{\sum_{i=1}^{n_k}\beta_i L_i I_i}{\sum_{i=1}^{n_k}\beta_i L_i}$$

Then

$$\oint ds\ \beta\left\langle\rho\right\rangle = \sum_{k=1}^{m} f_{0,k} w_k$$

in which the weight w_k is

$$w_k = \sum_{i=1}^{n_k}\beta_i L_i$$

Then we have

$$\overline{\left\langle\rho\right\rangle} = \frac{\sum_{k=1}^{m} f_{0,k} w_k}{\oint ds\ \beta}.$$

Summarizing this with a slight change in notation, we can write

$$\overline{\left\langle\rho\right\rangle} = \frac{\sum_{k=1}^{m}\left\langle\rho_k(\overline{I_k})\right\rangle w_k}{\oint ds\ \beta},$$

in which

$$\overline{I_k} = \frac{\sum_{i=1}^{n_k}\beta_i L_i I_i}{w_k}$$

5 Individual tune shifts

5.1 Derivation

The coherent tune shift is

$$\Delta Q_{x(y)} = \frac{e}{4\pi E}\oint ds\ \beta_{x(y)}\frac{\partial\bar{E}_{x(y)}}{\partial\Delta x(y)},$$

in which $E = m_0c^2\gamma$ is the beam energy. The beam is assumed to be a Gaussian with rms sizes σ_x, σ_y. In the x direction, the linear gradient of the electric field due to the cloud, averaged over the beam distribution, is

$$\frac{\partial\bar{E}_x}{\partial\Delta x} = \frac{e}{4\pi\varepsilon_0\sigma_x^2}\int_{-\infty}^{\infty}dx\ \int_{-\infty}^{\infty}dy\rho(x,y)\tilde{w}_x(x/\sqrt{2}\sigma_x, x/\sqrt{2}\sigma_y),$$

in which $\rho(x,y)$ is the 3-dimensional cloud number density (assumed to be uniform in the direction of the beam), and the weight function is

$$\tilde{w}_x(u,v) = \int_0^\infty dp\, \frac{\mathrm{e}^{-u^2/(1+p)-v^2/(1+p/r)}\left(1+p-2u^2\right)}{(1+p)^{5/2}(r+p)^{1/2}},$$

and $r = \sigma_y^2/\sigma_x^2$. In the y direction,

$$\frac{\partial \bar{E}_y}{\partial \Delta y} = \frac{e}{4\pi\varepsilon_0\sigma_x^2}\int_{-\infty}^{\infty} dx\, \int_{-\infty}^{\infty} dy \rho(x,y)\tilde{w}_y(x/\sqrt{2}\sigma_x, x/\sqrt{2}\sigma_y),$$

in which

$$\tilde{w}_y(u,v) = \int_0^\infty dp\, \frac{\mathrm{e}^{-u^2/(1+p)-v^2/(1+p/r)}\left(r+p-2ru^2\right)}{(r+p)^{5/2}(1+p)^{1/2}}.$$

Thus,

$$\Delta Q_x = \frac{r_e}{4\pi\sigma_x^2\gamma}\oint ds\, \beta_x \int_{-\infty}^{\infty} dx\, \int_{-\infty}^{\infty} dy\, \rho(x,y)\tilde{w}_x(x/\sqrt{2}\sigma_x, y/\sqrt{2}\sigma_y),$$

and

$$\Delta Q_y = \frac{r_e}{4\pi\sigma_x^2\gamma}\oint ds\, \beta_y \int_{-\infty}^{\infty} dx\, \int_{-\infty}^{\infty} dy\, \rho(x,y)\tilde{w}_y(x/\sqrt{2}\sigma_x, y/\sqrt{2}\sigma_y),$$

5.1.1 Tune shifts in terms of electric fields

The electric field is given by

$$\Omega_y(u,v) = \frac{1}{\sqrt{2}\sigma_x}\int_0^\infty dp \frac{v\sqrt{r}\mathrm{e}^{-u^2/(1+p)-rv^2/(p+r)}}{(r+p)^{3/2}(1+p)^{1/2}} = \frac{\tilde{\Omega}_y(u,v)}{\sqrt{2}\sigma_x}$$

so

$$\bar{E}_y(x_b,y_b) = \frac{e}{2\sqrt{2}\sigma_x\pi\varepsilon_0}\int_{-\infty}^{\infty} dx'\, \int_{-\infty}^{\infty} dy' \rho(x',y')\tilde{\Omega}_y((x_b-x')/(\sqrt{2}\sigma_x),(y_b-y')(\sqrt{2}\sigma_y))$$

The cloud density distribution is dependent on the beam position so we write

$$\bar{E}_y(x_b,y_b) = \frac{e}{2\sqrt{2}\sigma_x\pi\varepsilon_0}\int_{-\infty}^{\infty} dx'\, \int_{-\infty}^{\infty} dy' \rho(x',y',x_b,y_b)\tilde{\Omega}_y((x_b-x')/(\sqrt{2}\sigma_x),(y_b-y')(\sqrt{2}\sigma_y))$$

If we let

$$p_y(x_b,y_b) = \int_{-\infty}^{\infty} dx'\, \int_{-\infty}^{\infty} dy' \rho(x',y',x_b,y_b)\tilde{\Omega}_y((x_b-x')/(\sqrt{2}\sigma_x),(y_b-y')(\sqrt{2}\sigma_y)),$$

then

$$\bar{E}_y(x_b,y_b) = \frac{e}{2\sqrt{2}\sigma_x\pi\varepsilon_0}p_y(x_b,y_b)$$

The field gradient is

$$\frac{\partial \bar{E}_y}{\partial y_b} = \frac{1}{2\sqrt{2}\sigma_x\delta y_b\pi\varepsilon_0}\left(p_y(x_b,y_b+\delta y_b/2) - p_y(x_b,y_b-\delta y_b/2)\right)$$

The tune shift is

$$\Delta Q_y = \frac{e^2}{4\pi E}\frac{1}{2\sqrt{2}\sigma_x \delta y_b \pi\varepsilon_0}\oint ds\ \beta_y \left(p_y(x_b, y_b + \delta y_b/2) - p_y(x_b, y_b - \delta y_b/2)\right).$$

Using

$$r_e = \frac{e^2}{4\pi\varepsilon_0 m_0 c^2},$$

this can be written

$$\Delta Q_y = \frac{r_e}{2\sqrt{2}\pi\sigma_x \delta y_b \gamma}\oint ds\ \beta_y \left(p_y(x_b, y_b + \delta y_b/2) - p_y(x_b, y_b - \delta y_b/2)\right).$$

5.2 Evaluation

To compare with a simulation, we wish to evaluate

$$\oint ds\ \beta_i \left\langle \rho \right\rangle_{wx} = \overline{\left\langle \rho \right\rangle}_{wx} \oint ds\ \beta_i$$

in which $\left\langle \right\rangle_{wx}$ represents the density integrated with the weight function. There is a structurally similar expression for the y plane.

The ring is divided into n beamline elements, with index i. At each element, the radiation is characterized by the parameter I_i (photons per electron per meter, averaged over the element). The electron cloud density is a function of this parameter, and of the type of beamline element, designated by the index k (i.e., drift, dipole, wiggler, etc.): so at a given element

$$\left\langle \rho \right\rangle_{wx} = f_{x,k}(I_i)$$

If there are m types of beamline elements, and n_k elements of each type, then the integral around the ring can be written as

$$\oint ds\ \beta_x \left\langle \rho \right\rangle_{wx} = \sum_{k=1}^{m}\sum_{i=1}^{n_k} \beta_{x,i} L_i f_{x,k}(I_i).$$

If we make a simple linear approximation for the dependence of the density on the radiation parameter (an assumption which is likely to be wrong, given the nonlinear character of the electron cloud formation process), then we can write

$$f_{x,k}(I_i) = f_{x,0,k}\frac{I_i}{I_{x,0,k}}$$

in which $f_{x,0,k}$ is the weighted density corresponding to the radiation parameter $I_{x,0,k}$, and

$$\oint ds\ \beta_x \left\langle \rho \right\rangle_{wx} = \sum_{k=1}^{m}\frac{f_{x,0,k}}{I_{x,0,k}}\sum_{i=1}^{n_k} \beta_{x,i} L_i I_i$$

We can define the radiation parameter $I_{0,k}$ as being equal to the weighted average of I_i:

$$I_{x,0,k} = \frac{\sum_{i=1}^{n_k}\beta_{x,i} L_i I_i}{\sum_{i=1}^{n_k}\beta_{x,i} L_i}$$

Then

$$\oint ds\ \beta_x \left\langle \rho \right\rangle_{wx} = \sum_{k=1}^{m} f_{x,0,k} w_{x,k} = \sum_{k=1}^{m} \left\langle \rho(I_{x,0,k}) \right\rangle_{wx} w_{x,k}$$

in which the weight $w_{x,k}$ is

$$w_{x,k} = \sum_{i=1}^{n_k} \beta_{x,i} L_i$$

5.3 Evaluation

To compare with a simulation, we wish to evaluate

$$\Delta Q_{x(y)} = \frac{e}{4\pi E} \oint ds\ \beta_{x(y)} G_{x(y)} = \left\langle G_{x(y)} \right\rangle \oint ds\ \beta_{x(y)},$$

in which the field gradient is

$$G_{x(y)} = \frac{\partial \bar{E}_{x(y)}}{\partial x(y)}.$$

The ring is divided into n beamline elements, with index i. At each element, the radiation is characterized by the parameter I_i (photons per electron per meter, averaged over the element). The electron cloud density is a function of this parameter, and of the type of beamline element, designated by the index k (i.e., drift, dipole, wiggler, etc.): so at a given element

$$G_{x(y)} = g_{x(y),k}(I_i)$$

If there are m types of beamline elements, and n_k elements of each type, then the integral around the ring can be written as

$$\oint ds\ \beta_{x(y)} G_{x(y)} = \sum_{k=1}^{m} \sum_{i=1}^{n_k} \beta_{x(y),i} L_i g_{x(y),k}(I_i).$$

If we make a simple linear approximation for the dependence of the field gradient on the radiation parameter (an assumption which is likely to be wrong, given the nonlinear character of the electron cloud formation process), then we can write

$$g_{x(y),k}(I_i) = g_{x(y),0,k} \frac{I_i}{I_{x(y),0,k}}$$

in which $g_{x(y),0,k}$ is the field gradient corresponding to the radiation parameter $I_{x(y),0,k}$, and

$$\oint ds\ \beta_{x(y)} G_{x(y)} = \sum_{k=1}^{m} \frac{g_{x(y),0,k}}{I_{x(y),0,k}} \sum_{i=1}^{n_k} \beta_{x(y),i} L_i I_i$$

We can define the radiation parameter $I_{x(y),0,k}$ as being equal to the weighted average of I_i:

$$I_{x(y),0,k} = \frac{\sum_{i=1}^{n_k} \beta_{x(y),i} L_i I_i}{\sum_{i=1}^{n_k} \beta_{x(y),i} L_i}$$

Then

$$\Delta Q_{x(y)} = \frac{e}{4\pi E} \oint ds\ \beta_x G_{x(y)} = \frac{e}{4\pi E} \sum_{k=1}^{m} g_{x(y),0,k} w_{x(y),k}$$

in which the weight $w_{x,k}$ is

$$w_{x(y),k} = \sum_{i=1}^{n_k} \beta_{x(y),i} L_i$$

This can also be written as

$$\Delta Q_{x(y)} = \frac{e}{4\pi E} \sum_{k=1}^{m} G_{x(y),k}(\langle I_{x(y),k} \rangle) w_{x(y),k}$$

with

$$\langle I_{x(y),k} \rangle = \frac{\sum_{i=1}^{n_k} \beta_{x(y),i} L_i I_i}{w_{x(y),k}}$$