Sextupole magnet

Kai Hock

Cockcroft Institute / Liverpool University

June 22, 2010

The sextupole magnet is needed to correct for the chromatic effect of a quadrupole. Consider a quadrupole that focuses a beam of electrons in the x direction to a focal point. If some of the electrons have higher energies, they would be bent less by the quadrupole field, and their focal points would be further away, as illustrated in fig. 1. The result of this is that a bunch of electrons would get spread out or defocused in the longitudinal direction.

In order to correct for this chromatic effect, a sextupole magnet may be used. The job of a sextupole is illustrated in fig. 2. Compared with a quadrupole, a sextupole must have a larger focusing effect for particles that are displaced further from the axis. This happens with the sextupole because it is designed to have a magnetic field that varies as x^2, rather than the x^1 in the quadrupole.

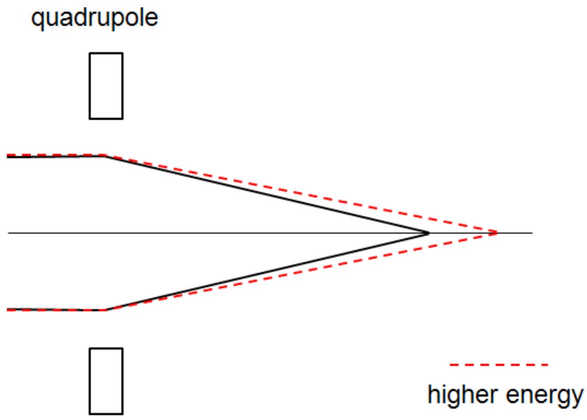

Figure 1: Chromatic effect.

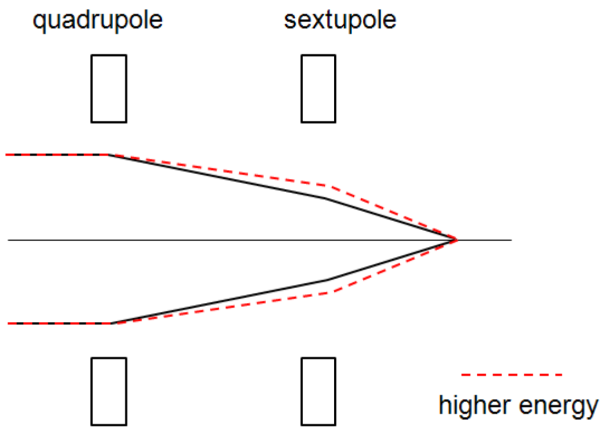

Figure 2: How a sextupole magnet works.

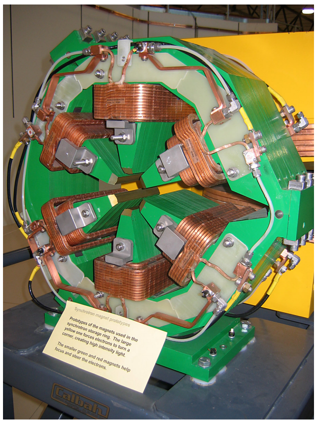

Figure 3: A sextupole magnet in Australian Synchrotron [1].

A photo of a real sextupole is shown in fig. 3. The field is shown schematically in fig. 4.

Here, I shall go into how the sextupole works in detail, but only explain how the sextupole field leads to the elements in the transfer matrix. The sextupole field is a bit more complex than the dipole and quadrupole fields. So it is convenient that there is one formula that can be used to derive all these fields. This is the multipole expansion.

We start with the magnetic scalar potential ψ, given by [2]:

$$\mathbf{B} = -\mu_0 \nabla \psi \tag{1}$$

where $\mathbf{B}$ is the magnetic field. For convenience, $\mu_0\psi$ will be replaced with ϕ. A multipole field expansion for the scalar field can be written as [6]:

$$\phi = \sum_{m=0}^{\infty} r^m \{A_m \cos m\theta + B_m \sin m\theta\} \tag{2}$$

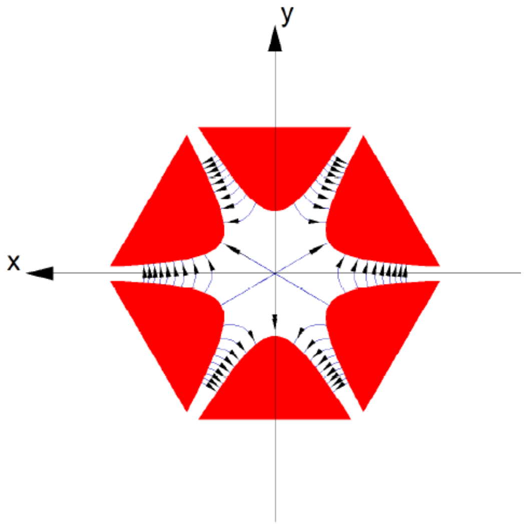

Figure 4: The field in a sectupole magnet [3].

where r is the distance from the axis, θ the azimuthal angle, and $\theta = 0$ is the positive x direction. A_m and B_m are constant coefficients. Lets look at the magnetic field arising from each term.

When $m = 0$, it is a constant term:

$$\phi = A_0 \tag{3}$$

so the field is zero:

$$\mathbf{B} = -\nabla\phi = 0\mathbf{i} + 0\mathbf{j} \tag{4}$$

When $m = 1$,

$$\phi = A_1 r \cos\theta + B_1 r \sin\theta \tag{5}$$

In Cartesian co-ordinates, it is:

$$\phi = A_1 x + B_1 y \tag{6}$$

So we get the dipole field:

$$\mathbf{B} = -\nabla\phi = -A_1\mathbf{i} - B_1\mathbf{j} \tag{7}$$

When $m = 2$:

$$\phi = A_2 r^2 \cos 2\theta + B_2 r^2 \sin 2\theta \tag{8}$$

which is the same as:

$$\phi = A_2(x^2 - y^2) + B_2 2xy \tag{9}$$

This gives the quadrupole field:

$$\mathbf{B} = -\nabla\phi = A_2(-2x\mathbf{i} + 2y\mathbf{j}) + B_2(-2y\mathbf{i} - 2x\mathbf{j}) \tag{10}$$

$A_2 = 0$ gives the normal quadrupole, and $B_2 = 0$ gives the skew quadrupole.

When $m = 3$:

$$\phi = A_3 r^3 \cos 3\theta + B_3 r^3 \sin 3\theta \tag{11}$$

Rewriting this in Cartesian co-ordinates, the sextupole can be derived:

$$\mathbf{B} = -\nabla\phi = A_3(-3(x^2 - y^2)\mathbf{i} + 6xy\mathbf{j}) + B_3(-6xy\mathbf{i} - 3(x^2 - y^2)\mathbf{j}) \tag{12}$$

For the normal sextupole, $A_3 = 0$, and the x and y magnetic field components are:

$$B_x = -6B_3 xy \tag{13}$$

$$B_y = -3B_3(x^2 - y^2) \tag{14}$$

The transfer matrix elements in [4] is written in terms of the second derivative of the field, so I shall do this now. Differentiating:

$$\frac{\partial^2 B_y}{\partial x^2} = -6B_3 \tag{15}$$

In terms of this:

$$B_x = \frac{\partial^2 B_y}{\partial x^2} xy \tag{16}$$

$$B_y = \frac{1}{2}\frac{\partial^2 B_y}{\partial x^2}(x^2 - y^2) \tag{17}$$

We are now ready to derive the elements for the sextupole. The sextupole is simply treated as a kicker. Thus, B_y produces a horizontal force and is modelled as a horizontal kicker. Likewise, the effect from B_x is modelled as a vertical kicker.

Then the elements for a kicker can be used directly, and the expressions for the sextupole field substituted.

The horizontal kicker elements are [5]:

$$m_{26} = -m_{51} = -m_{27} = \left(\frac{l}{B\rho}\right) B_y \tag{18}$$

$$m_{67} = -C_\gamma E_0^3 \left[\left(\frac{l}{B\rho}\right) B_y\right]^2 \left(\frac{1}{2\pi l}\right) \tag{19}$$

$$m_{27} = \frac{1}{2}\left[\left(\frac{l}{B\rho}\right) B_y\right] m_{67} \tag{20}$$

Substituting the expressions for B_y in eq. (17), we get:

$$m_{26} = -m_{51} = -m_{27} = \left(\frac{l}{B\rho}\right)\frac{1}{2}\frac{\partial^2 B_y}{\partial x^2}(x^2 - y^2) \tag{21}$$

$$m_{67} = -C_\gamma E_0^3 \Big[\Big(\frac{l}{B\rho}\Big)\frac{1}{2}\frac{\partial^2 B_y}{\partial x^2}(x^2-y^2)\Big]^2\Big(\frac{1}{2\pi l}\Big) \quad (22)$$

$$m_{27} = \frac{1}{2}\Big[\Big(\frac{l}{B\rho}\Big)\frac{1}{2}\frac{\partial^2 B_y}{\partial x^2}(x^2-y^2)\Big] m_{67} \quad (23)$$

Using the symbol defined in [4]:

$$\lambda = \frac{l}{B\rho}\frac{\partial^2 B_y}{\partial x^2} \quad (24)$$

we obtain the required sextupole transfer matrix elements:

$$m_{26} = -m_{51} = -m_{27} = \frac{1}{2}\lambda(x^2-y^2) \quad (25)$$

$$m_{67} = -C_\gamma E_0^3 \Big[\frac{1}{2}\lambda(x^2-y^2)\Big]^2\Big(\frac{1}{2\pi l}\Big) \quad (26)$$

$$m_{27} = \frac{1}{2}\Big[\frac{1}{2}\lambda(x^2-y^2)\Big] m_{67} \quad (27)$$

The element m_{27} is not given in [4], possibly because it is small.

The values of x and y just before the sextupole are x_0 and y_0, so:

$$m_{26} = -m_{51} = -m_{27} = \frac{1}{2}\lambda(x_0^2-y_0^2) \quad (28)$$

$$m_{67} = -C_\gamma E_0^3 \Big[\frac{1}{2}\lambda(x_0^2-y_0^2)\Big]^2\Big(\frac{1}{2\pi l}\Big) \quad (29)$$

So far, we have derived the matrix elements for the effects of B_y using the formulae for the horizontal kicker. The matrix elements for the effects of B_x can be derived similarly, using the formulae for the vertical kicker [4], and the expression for B_x in eq. (17). The first few elements, for example, are given by:

$$-m_{46} = m_{53} = m_{47} = \lambda x_0 y_0 \quad (30)$$

$$m_{67} \;=\; -C_\gamma E_0^3\Big[\lambda x_0 y_0\Big]^2\left(\frac{1}{2\pi l}\right) \tag{31}$$

Note that there is a common term m_{67} in eqs. (29) and (31). They are separate contributions from B_y and B_x respectively. So the combined effect would be the sum. Adding, we get:

$$m_{67} = -C_\gamma E_0^3\lambda^2(x_0^2 + y_0^2)^2\left(\frac{1}{8\pi l}\right) \tag{32}$$

References

[1] http://en.wikipedia.org/wiki/Sextupole_magnet

[2] http://en.wikipedia.org/wiki/Magnetic_potential

[3] Andy Wolski's linear dynamics lecture 3, http:http://hep.ph.liv.ac.uk/~hock/Good_References/Good_References.html

[4] Alex Chao, Evaluation of Beam Distribution Parameters in an Electron Storage Ring, SLAC-PUB-2143. http://www.slac.stanford.edu/pubs/slacpubs/2000/slac-pub-2143.html

[5] http://hep.ph.liv.ac.uk/~hock/Damping_Ring/kicker_magnet.html

[6] P. J. Bryant and K. Johnsen, Circular Accelerators and Storage Rings, (Cambridge University Press, UK) 1993, pp. 317 - 319.