Modeling Horizontal Beam Size Calculations Using Sextupoles

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I present here a summary of the method described in [1] to measure particle beam size using sextupole magnets, and a verification of the derivation's validity. I model the Cornell Electron Storage Ring with toy lattices in the Bmad library, and implement the derivation as a means of justifying its approximations. Results are shown for lattices with 1, 2, 4, and 25 sextupoles, with sextupole misalignments ranging from $X_{mis} = 0$ to 2.7 mm, and for sextupole strength changes ranging from $K_2 = -5$ to 5 m^{-3}. Using orbit data to calculate the beam size works extremely well regardless of the number of sextupoles in the ring or the misalignment of the measured sextupole (MAPE .02% SD .05%). Using tune data works well for rings with a single sextupole (MAPE 1.8% SD 5.6%), and for rings with additional sextupoles with strengths small compared to the measured sextupole (MAPE 6.9% SD 38.3%). The error grows in rings with multiple sextupoles with large K_2 values (MAPE 23.4% SD 51.7%). This error is heightened when the initial misalignment of the measured sextupole is greater, and when the sextupole strength change approaches zero. The approximations made in deriving the IPAC '21 calculation accurately measure beam size, with systematic error tracing to a few key assumptions.

I. BACKGROUND

The goal of my master's project is to determine the validity of the derivation presented in [1], a method of measuring beam size with sextupole magnets. I do this by simulating the Cornell Electron Storage Ring with toy lattices in the Bmad subroutine library [2] visualized by the Tao user interface [3]. I then calculate the beam size of my simulations using the calculation derived in [1], and compare the computed result with that expected based on input parameters. An overview of this derivation is presented here, along with the details of my model rings and methods of simulation. I then describe the results from implementing the calculation as they compare to theory, and discuss sources of error as well as opportunities for further research. Previous work has found that this method of changing the sextupole strength provides accurate data about ring optics [4], and agrees with other methods of measuring the beam size [5]. The degree of accuracy in this calculation implemented in a toy lattice justifies the validity of the assumptions made in the derivation, and ultimately supports the feasibility of implementing this technique in CESR and other accelerator rings.

II. THEORY

As a particle beam traverses a sextupole magnet, it receives a beam-size dependent kick due to the quadratic dependence of the magnet's field on position [1]. This kick is decomposed into a dipole term ($\Delta X'$) and a quadrupole term (ΔK_1) [6]. We use the kick at two different values of sextupole strength to measure the size of the beam. This procedure is greatly simplified by setting the initial sextupole strength to zero.

The angle change through the magnet at a specific sextupole strength K_2 is found by integrating the force of the magnet on the beam over the length of the sextupole. This force is calculated by integrating the Lorentz force over the transverse extent of the beam.

$$X' = \frac{1}{cP}\int_0^L F ds \tag{1}$$

$$F = \frac{qcB_y''}{\sigma\sqrt{8\pi}}\int_{-\infty}^{+\infty} x^2 \exp\left(\frac{-(x-X_0)^2}{2\sigma^2}\right)dx \tag{2}$$

$$\text{where } B_y = \frac{K_2 P}{2q}x^2$$

$$F = \frac{1}{2}cPK_2(X_0^2+\sigma^2)$$

$$\implies X' = \frac{1}{2}K_2 L(X_0^2+\sigma^2) \tag{3}$$

Where P is the longitudinal momentum of the beam, q is the fundamental charge of a single particle, L is the length of the sextupole, X_0 is the distance between the centroid of the beam and the center of the sextupole, K_2 is the sextupole strength, and σ is the size of the beam (i.e. the standard deviation of a distribution of particles in equation (2)). For a full derivation of equation (1) see [7].

When the sextupole strength changes, so does the angle change that the particles experience as they traverse

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the magnet. At a final sextupole strength of $K_2+\Delta K_2$, the final angle change $(X'+\Delta X')$ through the sextupole is:

$$X' + \Delta X' = \frac{1}{2}(K_2 + \Delta K_2)L((X_0 + \Delta X)^2 + \sigma^2) \quad (4)$$

Where $K_2 = K_{2,Initial}$ is the initial strength of the sextupole.

Subtracting (3) from (4) results in the difference in angle change before and after the sextupole strength is varied.

$$\begin{aligned}\Delta X' &= \frac{1}{2}\Delta K_2 L((X_0 + \Delta X)^2 + \sigma^2) \\ &+ \frac{1}{2}K_2 L(2X_0\Delta X + \Delta X^2)\end{aligned} \quad (5)$$

Setting $K_2 = 0$, a prudent choice where the sextupole is initially turned off, and solving for σ results in an equation for the beam size calculated from orbit data:

$$\sigma^2 = 2\frac{\Delta X'}{\Delta K_2 L} - (X_0 + \Delta X)^2 \quad (6)$$

The quadrupole term can then be used to solve for $X_0 + \Delta X$. A sextupole strength K_2 provides a quadrupole kick K_1:

$$K_1 = K_2 X_0 \quad (7)$$

Now we perform the same procedure as above of varying K_2, which changes both the orbit and the quadrupole term:

$$K_1 + \Delta K_1 = (K_2 + \Delta K_2)(X_0 + \Delta X) \quad (8)$$

Subtracting equation (7) from (8) and again setting $K_2 = K_{2,Initial} = 0$ results in a relationship between the quadrupole error and the orbit data:

$$\frac{\Delta K_1}{\Delta K_2} = (X_0 + \Delta X) \quad (9)$$

Substituting equation (9) in equation (6) results in an equation for beam size using the quadrupole kick, rather than the orbit:

$$\sigma^2 = 2\frac{\Delta X'}{\Delta K_2 L} - (\frac{\Delta K_1}{\Delta K_2})^2 \quad (10)$$

This is completed by the relationship between the change in tune (ΔQ) and the change in the quadrupole term. If M is the one turn matrix around the ring, then $Tr(M) = 2\cos(2\pi Q)$. Denoting the transfer matrix for a quadrupole of infinitesimal length as M_Q and for the rest of the ring as M_R, the one turn matrix can be decomposed into $M = M_Q M_R$. Introducing a quadrupole error of ΔK_1 changes matrix M_Q into M_Q^*:

$$\begin{aligned}M_Q = \begin{bmatrix} 1 & ds \\ -K_1 ds & 1 \end{bmatrix} &\rightarrow M_Q^* = \begin{bmatrix} 1 & ds \\ -(K_1 + \Delta K_1)ds & 1 \end{bmatrix} \\ \implies M_Q^* &= \begin{bmatrix} 1 & 0 \\ -\Delta K_1 ds & 1 \end{bmatrix} M_Q\end{aligned}$$

The resultant perturbation in the one turn matrix can be expressed as

$$M^* = \begin{bmatrix} 1 & 0 \\ -\Delta K_1 ds & 1 \end{bmatrix} M$$

and the new one turn matrix M^* still obeys $Tr(M^*) = 2\cos(2\pi(Q) + \Delta Q))$, leading to a relationship between tune change and quadrupole error:

$$\begin{aligned}2\cos(2\pi Q) - \Delta K_1 \beta L \sin(2\pi Q) \\ = 2\cos(2\pi(Q + \Delta Q))\end{aligned} \quad (11)$$

For small changes in tune, this can be approximated as:

$$\Delta K_1 = \frac{4\pi\Delta Q}{\beta L} \quad (12)$$

For a full derivation of equation (12) see [6]. Using Equations (10) and (12), the change in tune can be used with the initial beta function and the difference in angle change to find the beam size.

III. IMPLEMENTATION

To implement the theory discussed in the previous section, I created four lattice files in Bmad, each representing a ring with a different total number of sextupoles (1, 2, 4, and 25). Each lattice file has a total circumference of 775.2 m (roughly the same size as CESR) composed of 51 FODO cells, with sextupoles superimposed upon the drift regions of FODO cells. The measured sextupoles have length .02 m (subtended angle of 4.05×10^{-5} rad) to accurately model the thin lens approximation used in the derivations. The strengths of all the sextupoles in the lattice were first optimized to ensure a chromaticity of 0, and a single sextupole was subsequently turned off entirely. This initialization determined the strengths of the additional sextupoles in the ring, and ensured that $K_{2,Initial} = 0$ for the sextupole with which the analysis was performed (labeled here "the measured sextupole"). The initial tune of all rings before the sextupole strength change was $Q_0 = 11.893$ and the initial beta function directly preceding the measured sextupole was $\beta_0 = 23.359\ m$ for lattices with 1 and 2 sextupoles, and $\beta_0 = 5.569\ m$ for lattices with 4 and 25 sextupoles.

Additional sextupoles in the ring also have the potential to affect the analysis. Varying the strength of the measured sextupole changes the closed orbit around the entire ring, including in the additional sextupoles, which have a quadratic dependence of field on position not accounted for in the derivation above. Table I displays the strengths of the additional sextupoles in the ring. The strengths of additional sextupoles decreases as the total number of sextupoles in the ring increases.

Total number of sextupoles	$\|K_2\|$ of additional sextupoles (m^{-3})
1	–
2	45.06
4	12.24
25	.98

Table I. Additional sextupole strengths for four toy lattices.

After initialization, I add a misalignment (X_{mis}) of anywhere between 0 and 2.7 mm to the measured sextupole. This misalignment does not affect the orbit, as the sextupole is still turned off ($K_{2,Initial} = 0$). Because the reference orbit is defined by the dipoles in the lattice, which have not been moved, the distance between the centroid of the beam and the center of the sextupole is equal to the inputted misalignment: $X_0 = -X_{mis}$.

I then change the sextupole strength from its initial value of 0 to an integer multiple of .5 in the range $[-5, 5]\ m^{-3}$. At this point, I add an explicit dipole kick

$$b_0 = \frac{1}{2}\Delta K_2 L \sigma_{in}^2 \tag{13}$$

to the sextupole for a typical value of σ_{in} (about 2 mm). This dipole kick acting on a centroid simulates the dipole kick that a full beam experiences as it traverses a magnet. The goal of the calculations presented in part II is to use equation (10) to solve for the beam size inputted as a parameter in equation (13). The analysis presented here attempts to recreate the horizontal beam size only.

IV. RESULTS

Key results are presented for both single and multi sextupole lattices. Many of the results shown here calculate beam size as a function of varying either σ_{in}, X_{mis}, or ΔK_2 while holding the other two values constant.

A. Single Sextupole Lattice

For a lattice with a single sextupole initially turned off, figure 1 shows the result of calculating the beam size using orbit data (i.e. using equation (6)) vs. the expected beam size inputted in equation (13). The mean absolute error (MAE) in the beam size when calculated from the orbit is $4.2 \times 10^{-5}\ mm$ with a standard deviation (SD) of $4.2 \times 10^{-4}\ mm$. This small error, along with the linear relationship of figure 1, shows that equation (6) is a very good approximation to the beam size in a ring with a single sextupole that is initially off, even with a non-zero misalignment. This result persists for different values of sextupole strength changes (both positive and negative), and for different values of misalignments.

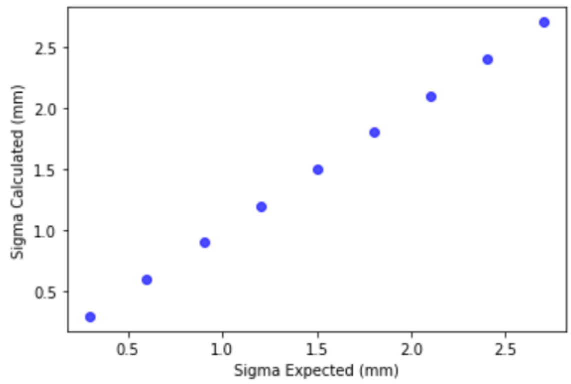

Figure 1. Calculated vs. Expected beam size using orbit data, for a constant value of $X_{mis} = 1.2\ mm$ and $\Delta K_2 = 3\ m^{-3}$.

When using the quadrupole kick to calculate the beam size, i.e. calculating the beam size from equation (10), the relationship between expected and calculated sigma stays linear (figure 2). The MAE in the beam size when calculated from the tune is $5.0 \times 10^{-3}\ mm$ with SD $2.9 \times 10^{-3}\ mm$. Although figure 2 looks identical to figure 1, the error increases by two orders of magnitude when performing the calculation using the tune data. As the change in sextupole strength decreases ($|\Delta K_2| \to 0$), this increased error becomes more pronounced. For a constant beam size of $\sigma_{in} = 2.1\ mm$ and constant misalignment of $X_{mis} = 1.2\ mm$, the error in σ_{calc} reaches its maximum absolute value of $2.22 \times 10^{-2}\ mm$ at the smallest ΔK_2 scanned, equal to $+.5\ m^{-3}$. However, even this maximum error is only about 1% of the beam size, leading us to conclude that using tune data is also an accurate method of measuring beam size for the single sextupole case.

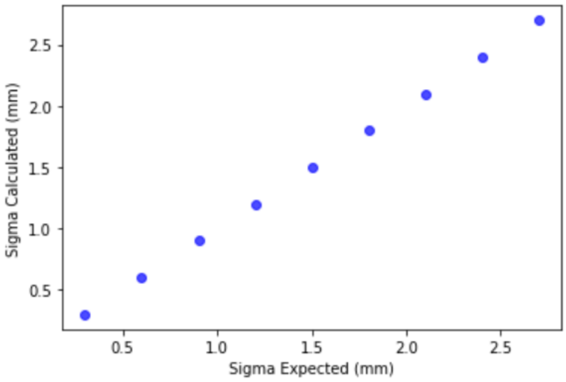

Figure 2. Calculated vs. Expected beam size using tune data, for a constant value of $X_{mis} = 1.2\ mm$ and $\Delta K_2 = 3\ m^{-3}$.

B. Multiple Sextupole Lattices

The analysis reported above is now repeated with lattices that contain additional sextupoles around the ring. The additional sextupoles remain unvaried through the procedure. Results for calculated beam size as a function of expected beam size are shown in figure 3.

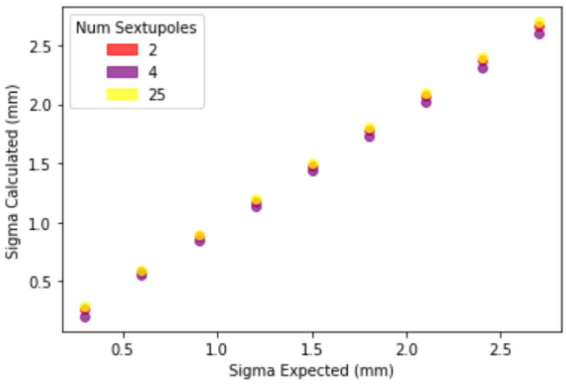

Figure 3. Calculated vs. Expected beam size using tune data, for a constant value of $X_{mis} = 1.2\ mm$ and $\Delta K_2 = 3\ m^{-3}$. Different colors represent data for lattices with different numbers of sextupoles.

The mean absolute error and standard deviation are shown in table II. In all cases with multiple sextupoles, the MAE and SD are at least one order of magnitude larger than in the case of a single sextupole, suggesting that using this technique in multi sextupole rings is less precise than in a single sextupole ring.

The beam size calculated from tune data aligns well with the expected beam size for constant values of ΔK_2 and X_{mis}. This linearity is reproduced in plots with different values of ΔK_2 (both positive and negative) and small values of X_{mis} relative to the beam size.

Number of sextupoles	MAE (mm)	SD (mm)
1	5.0×10^{-3}	2.9×10^{-3}
2	4.6×10^{-2}	1.8×10^{-2}
4	1.3×10^{-1}	2.8×10^{-2}
25	3.2×10^{-2}	1.4×10^{-2}

Table II. Mean error and standard deviation when calculating beam size using tune data for multi-sextupole lattices.

When ΔK_2 and σ_{in} are held constant, the error in the calculated beam size grows as a function of increasing misalignment (figure 4). The error becomes increasingly pronounced as the misalignment in the sextupole reaches values comparable to the size of the beam. For $\Delta K_2 = 3\ m^{-3}$ and $\sigma_{in} = 2.1\ mm$, beam size error reached its maximum values at the largest value of sextupole misalignment tested ($2.7\ mm$). The maximum absolute error in the lattice with 2 sextupoles is $.19\ mm$ (8.9%) while in the lattice with 4 sextupoles it is $.63\ mm$ (29.8%), and in the lattice with 25 sextupoles it is $.12\ mm$ (5.5%). It remains unclear why the error for various misalignments is always negative, but this phenomenon persists regardless of the K_2 change in the analysis. Most notably, this is true for both positive and negative values of ΔK_2. This indicates that equation (10) underestimates the size of the beam. Negative misalignment values were not tested.

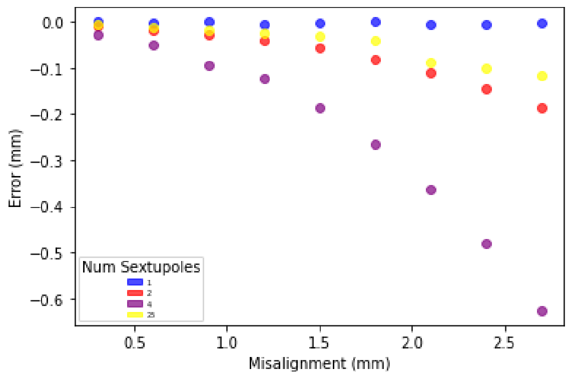

Figure 4. Error in beam size calculation as a function of misalignment, for multiple sextupoles in a ring. Constant $\Delta K_2 = 3\ m^{-3}$ and $\sigma_{in} = 2.1\ mm$.

In lattices where the additional sextupoles are strong relative to the measured sextupole, the error in the calculated beam size increases dramatically. Figure 5 shows a snapshot of figure 4 when the misalignment is held constant at $X_{mis} = .9\ mm$, and displays error as a function of ΔK_2. In the case of a single sextupole or many relatively weak sextupoles, the error only increases near $\Delta K_2 = 0$, with average values of .20% and 1.2%. In the case of the two and four sextupole lattices the error in the beam size calculations are higher, with averages 2.6% and 5.7% respectively.

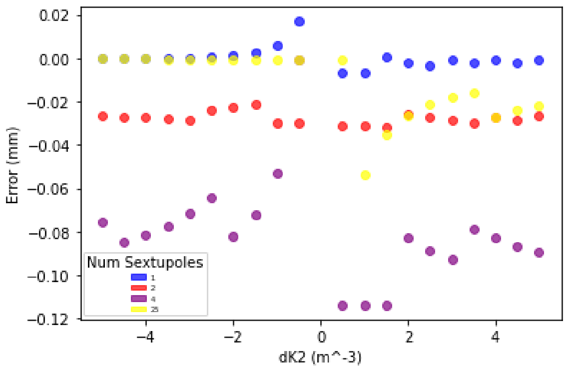

Figure 5. Error in beam size calculation as a function of sextupole strength change, for multiple sextupoles in a ring. Constant $X_{mis} = .9\ mm$ and $\sigma_{in} = 2.1\ mm$.

Figure 4 shows that large misalignments adversely influence the beam size calculation, especially for lattices with additional sextupoles that are strong relative to the measure sextupole. Meanwhile, figure 5 shows that while error is larger for lattices with stronger additional sextupoles, the beam size calculation is not impacted by the choice of ΔK_2 used to perform the analysis.

C. Validating Approximations

In order to check the approximations made in the derivation in part II, we plot exact vs. approximate values for two important quantities. The first is to verify that equation (12) is a sufficient approximation for equation (11) using our values for tune change. Figure 6 plots ΔK_1 as computed from the small tune change approximation vs. the exact value, for many different values of ΔK_2. This plot is representative of others with different values of X_{mis} and σ_{in}. The MAE in the approximation is $1.4 \times 10^{-6}\ m^{-2}$ (.03%) with SD $1.6 \times 10^{-6}\ m^{-2}$ (.03%). The approximation is therefore valid in the regime of tune changes which result from the whole range of sextupole strength changes tested.

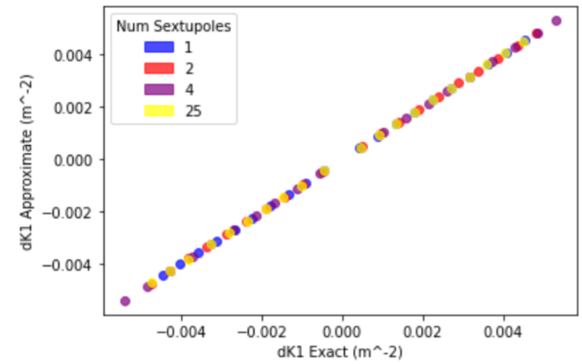

Figure 6. ΔK_1 Approximate vs. ΔK_1 Exact, for a constant value of $\sigma_{in} = 2.1\ mm$ and a constant $X_{mis} = .9\ mm$.

An additional error stems from equation (7) which leads to the relationship of equation (9), allowing us to use tune data rather than orbit data to complete the beam size measurement. Figure 7 plots the misalignment as calculated by reforming equation (9) into: $X_0 = \frac{\Delta K_1}{\Delta K_2} - \Delta X$ for different values of initial misalignment.

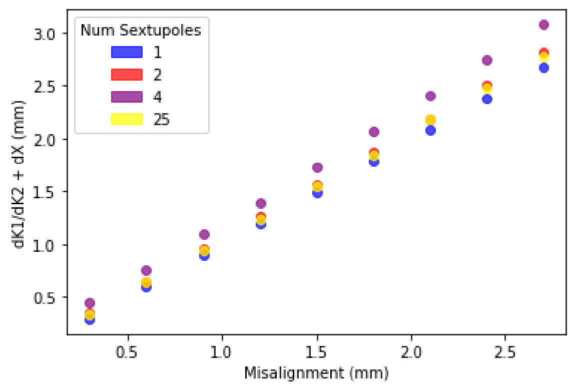

Figure 7. X_0 evaluated using tune data vs. initial misalignment of sextupole, for constants values of inputted beam size $\sigma_{in} = 2.1\ mm$ and sextupole strength change $\Delta K_2 = 3\ m^{-3}$.

The linearity shows relative agreement between tune and orbit data, but the vertical spread at each value of misalignment shows error in computing the X_0 from the tune. In all lattices with multiple sextupoles, the tune overestimates the misalignment. Higher purple and red dots indicate that this problem is exacerbated in the 2 and 4 sextupole lattice, with MAE 7.17 SD $2.24 \times 10^{-2}\ mm$ (6.1%) and MAE 2.45 SD $.79 \times 10^{-1}\ mm$ (20.6%) respectively. This overestimation of the misalignment implies that ΔK_1 is overestimated by equation (7), and is the most likely cause of underestimation in the calculated beam size when using equation (10).

V. CONCLUSION

The analysis presented here describes the validity of using equation (10) to measure the beam size through a sextupole. When using orbit data, the beam size can be calculated to within .1%, regardless of the initial misalignment of the sextupole or the additional sextupoles in the lattice. When using tune data, the error in the calculation of the beam size stays within 1% of the expected value for a lattice with a single sextupole (SD 5.6%). However, when additional sextupoles are present in the ring, the error rises dramatically to an average absolute value of 24.8% (SD 362.7%) in a ring with 2 sextupoles, 23.4% (SD 51.7%) in a ring with 4 sextupoles, and 6.9% (SD 38.3%) in a ring with 25 sextupoles. This increased error is not due to the approximation of small tune change simplifying equation (11) into equation (12). Rather the change in the quadrupole term ΔK_1 is calculated to be larger than expected when using equations (7) and (8).

Additional sextupoles in a ring lead to increased error based on two parameters: the number of additional sextupoles and their strengths. The greatest error was found in the ring with 4 sextupoles due to their large K_2 values. Meanwhile, the ring with 25 sextupoles did not exhibit much error in beam size measurements as the additional magnets were weak relative to the measured sextupole.

Although the analysis presented here is for a short sextupole of length .02 m, many of the plots show similar trends when using a sextupole of length .3 m. This longer sextupole is still short when compared to the total circumference of the ring (subtended angle .0024 rad), but more accurately represents the length of a typical sextupole in CESR.

Additionally, the entire analysis tends to break down at smaller sextupole strength changes. The smallest K_2 changes that were analyzed led to the greatest error. This result persisted across all lattices.

This modeling campaign supports experimental measurements illustrated in [8], which analyze sources of error when performing this beam size calculation technique in CESR.

Future research attempting to measure beam size with sextupole strength changes should continue to analyze sources of error, especially in equation (7). One method of mitigating the error found in lattices with multiple sextupoles is to include corrector coils before and after the measured sextupole in order to negate the effects of the changing sextupole strength on orbit. The next step in the simulation is to perform the same analysis on a lattice which more adequately models CESR, in both the x and y dimensions, and beginning at a non-zero sextupole strength.

VI. ACKNOWLEDGMENTS

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