July 15, 2021
JAC

Uncertainty Propagation for Beam Size Determination Using Sextupoles

Assuming the two terms are independent,

$$\boldsymbol{\sigma^2 = 2\frac{\mathrm{d}x'}{\mathrm{d}k_2 l} - \left(\frac{\mathrm{d}k_1 l}{\mathrm{d}k_2 l}\right)^2},$$

$$\boldsymbol{\left(\delta\sigma^2\right)^2 = \left(\delta\left(2\frac{\mathrm{d}x'}{\mathrm{d}k_2 l}\right)\right)^2 + \left(\delta\left(\frac{\mathrm{d}k_1 l}{\mathrm{d}k_2 l}\right)^2\right)^2}$$

$$\boldsymbol{\left(2\ \sigma\delta\sigma\right)^2 = \left(2\ \delta\left(\frac{\mathrm{d}x'}{\mathrm{d}k_2 l}\right)\right)^2 + \left(2\ \frac{\mathrm{d}k_1 l}{\mathrm{d}k_2 l}\ \delta\left(\frac{\mathrm{d}k_1 l}{\mathrm{d}k_2 l}\right)\right)^2}$$

$$\boldsymbol{\left(\delta\sigma\right)^2 = \frac{1}{4\ \sigma^2}\left(\left(2\ \delta\left(\frac{\mathrm{d}x'}{\mathrm{d}k_2 l}\right)\right)^2 + \left(2\ \frac{\mathrm{d}k_1 l}{\mathrm{d}k_2 l}\ \delta\left(\frac{\mathrm{d}k_1 l}{\mathrm{d}k_2 l}\right)\right)^2\right)}$$

Example: IPAC21 (Sextupole 10aw) wave analyses assuming second-order effects are small

$$\mathbf{x_0 = 5.262 \pm 0.026\ mm\ (0.5\%)}$$

$$\boldsymbol{\frac{\mathrm{d}x'}{\mathrm{d}k_2 l} = \mathbf{14.27 \pm 0.14}\ \mu\mathbf{rad/m^{-2}\ (1.0\%)}}$$

$$\boldsymbol{\sigma^2 = \mathbf{2 \times 14.27 - (5.262)^2 = 28.54 - 27.69 = 0.85\ mm^2}}$$

$$\boldsymbol{\sigma = \mathbf{0.92\ mm}}$$

$$\boldsymbol{\left(\delta\sigma\right)^2 = \frac{1}{4(0.85)}\left((2\ (0.14))^2 + (2\ (5.262)(0.026))^2\right)\ \mathrm{mm}^2}$$

$$\boldsymbol{\left(\delta\sigma\right)^2 = \frac{1}{4(0.85)}\left(0.078 + 0.075\ \right) = 0.045\ \mathrm{mm}^2}$$

$$\boldsymbol{\delta\sigma = \mathbf{0.21\ mm}}$$

A 3.0% effect in the difference of squares is measured with an accuracy of 45%, so the error in $\boldsymbol{\sigma}$ is 23%.

Example: IPAC21 (Sextupole 10aw)
Orbit wave analysis and quad kick from tunes
assuming second-order effects are small

$$\mathrm{x}_0 = 5.126 \pm 0.013 \text{ mm } (0.3\%)$$

$$\frac{\mathrm{d}x'}{\mathrm{d}k_2 l} = 14.27 \pm 0.14 \ \mu\text{rad/m}^{-2} \ (1.0\%)$$

$$\sigma^2 = 2 \times 14.27 - (5.126)^2 = 28.54 - 26.28 = 2.26 \text{ mm}^2$$

$$\sigma = 1.50 \text{ mm}$$

$$(\delta\sigma)^2 = \frac{1}{4(2.26)} \left((2\ (0.14))^2 + (2\ (5.126)(0.013))^2\right) \text{ mm}^2$$

$$(\delta\sigma)^2 = \frac{1}{4(2.26)} (0.078 + 0.018\ ) = 0.011 \text{ mm}^2$$

$$\delta\sigma = 0.10 \text{ mm}$$

An 8% effect in the difference of squares is measured with an accuracy of 13%, so the error in $\boldsymbol{\sigma}$ is 7%.

Example: IPAC21 (Sextupole 10aw)
Modeled orbit change from CesrV using measured orbit and quad kick from wave analysis assuming second-order effects are small

$$\mathbf{x_0 = 5.262 \pm 0.026\ mm\ (0.5\%)}$$

$$\begin{aligned}
\frac{\mathrm{d}x'}{\mathrm{d}k_2 l} &= \frac{2\tan(\pi Q)}{\beta}\frac{\mathrm{d}x}{\mathrm{d}k_2 l} \\
&= 0.287\ \frac{\mathrm{d}x}{\mathrm{d}k_2 l} \\
&= 0.287\ (50.4 \pm 1.4) = 14.46 \pm 0.40\ \mu\mathrm{rad/m}^{-2}\ (2.2\%)
\end{aligned}$$

$$\sigma^2 = 2 \times 14.46 - (5.262)^2 = 28.92 - 27.69 = 1.23\ \mathrm{mm}^2$$
$$\sigma = 1.11\ \mathrm{mm}$$

$$(\delta\sigma)^2 = \frac{1}{4(1.23)}\left((2\ (0.40))^2 + (2\ (5.262)(0.026))^2\right)\ \mathrm{mm}^2$$
$$(\delta\sigma)^2 = \frac{1}{4(1.23)}(0.16 + 0.075\ )\ \mathrm{mm}^2$$

$$\delta\sigma = 0.22\ \mathrm{mm}$$

A 4.3% effect in the difference of squares is measured with an accuracy of 40%, so the error in $\boldsymbol{\sigma}$ is 20%.

Example: Sextupole 10aw
Single-kick analysis for orbit kick
and quad kick from tunes (IPAC21)
assuming second-order effects are small

$$\mathbf{x_0 = 5.126 \pm 0.013\ mm\ (0.3\%)}$$

$$\frac{\mathrm{d}\boldsymbol{x}'}{\mathrm{d}\boldsymbol{k_2 l}} = \mathbf{13.71 \pm 0.12\ \mu rad/m^{-2}\ (0.8\%)}$$

$$\boldsymbol{\sigma}^2 = \mathbf{2 \times 13.71 - (5.126)^2 = 27.41 - 26.28 = 1.13\ mm^2}$$

$$\boldsymbol{\sigma} = \mathbf{1.06\ mm}$$

$$(\boldsymbol{\delta\sigma})^2 = \frac{\mathbf{1}}{\mathbf{4(1.13)}}\left(\mathbf{(2\ (0.12))^2 + (2\ (5.126)(0.013))^2}\right)\ \mathbf{mm^2}$$

$$(\boldsymbol{\delta\sigma})^2 = \frac{\mathbf{1}}{\mathbf{4(1.13)}}\ \mathbf{(0.058 + 0.018\ ) = 0.017\ mm^2}$$

$$\boldsymbol{\delta\sigma} = \mathbf{0.13\ mm}$$

A 4% effect in the difference of squares is measured with an accuracy of 28%, so the error in $\boldsymbol{\sigma}$ is 14%.