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Tune Shifts from Skew Quad Errors

Following Handbook 4.5.4 (Rubin), perturb the one-turn matrix at a place where $\boldsymbol{\alpha = 0}$:

$\boldsymbol{F} =$

$$\begin{bmatrix} \cos 2\pi Q_x & \beta_x \sin 2\pi Q_x & 0 & 0 \\ -\gamma_x \sin 2\pi Q_x & \cos 2\pi Q_x & 0 & 0 \\ 0 & 0 & \cos 2\pi Q_y & \beta_y \sin 2\pi Q_y \\ 0 & 0 & -\gamma_y \sin 2\pi Q_y & \cos 2\pi Q_y \end{bmatrix}$$

Thin skew quad Handbook 4.54 Eq. 5

$\boldsymbol{\Delta KL}$ is the strength of a normal quad which is now rotated by $\boldsymbol{45^\circ}$ to make the skew quad error.

$\mathbf{M_{thin}} =$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \Delta KL & 0 \\ 0 & 0 & 1 & 0 \\ \Delta KL & 0 & 0 & 1 \end{bmatrix}$$

Define

$$\begin{bmatrix} I & K_t \\ K_t & I \end{bmatrix}$$

Perturbed 1-turn matrix:

$\boldsymbol{P} = \boldsymbol{F}\ \mathbf{M_{thin}} =$

$$\begin{bmatrix} M & MK_t \\ NK_t & N \end{bmatrix}$$

So, $\mathbf{m} \equiv \boldsymbol{NK_t} =$

$$\begin{bmatrix} \Delta KL\, \beta_y \sin 2\pi Q_y & 0 \\ \Delta KL\ \cos 2\pi Q_y & 0 \end{bmatrix}$$

and, $\mathbf{n} \equiv \boldsymbol{MK_t} =$

$$\begin{bmatrix} \Delta KL\, \beta_x \sin 2\pi Q_x & 0 \\ \Delta KL\ \cos 2\pi Q_x & 0 \end{bmatrix}$$

The symplectic conjugate (Sagan and Rubin (PRSTAB **2**,074001 (1999)), Eq. 5)
$\mathbf{n}^+ =$

$$\begin{bmatrix} 0 & 0 \\ -\Delta KL\ \cos 2\pi Q_x & \Delta KL\, \beta_x \sin 2\pi Q_x \end{bmatrix}$$

and $\mathbf{m} + \mathbf{n}^+ =$

$$\begin{bmatrix} \Delta KL\, \beta_y \sin 2\pi Q_y & 0 \\ \Delta KL\ (\cos 2\pi Q_y - \cos 2\pi Q_x) & \Delta KL\, \beta_x \sin 2\pi Q_x \end{bmatrix}$$

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Handbook 4.54 Eq. 28

The tune split due to coupling is given by:

$$\begin{aligned} \mathrm{Tr}\left(A-B\right) &= 2\left(\cos 2\pi Q_A - \cos 2\pi Q_B\right) \\ &= \sqrt{\mathrm{Tr}\left(M-N\right)^2 + 4\det\left(\mathrm{m}+\mathrm{n}^+\right)} \\ &= \sqrt{4\left(\cos 2\pi Q_x - \cos 2\pi Q_y\right)^2 + 4\left(\Delta KL\right)^2 \beta_x\beta_y \sin 2\pi Q_x \sin 2\pi Q_y} \end{aligned}$$

Special case $Q_x = Q_y$

For the case $Q_x = Q_y$, the tune split is symmetric.
Defining $\Delta Q_{AB} \equiv \left(Q_A - Q_B\right)/2$, we have $Q_A = Q_x + \Delta Q_{AB}$ and $Q_B = Q_y - \Delta Q_{AB}$.
Also define $\mu \equiv 2\pi Q_x = 2\pi Q_y$ and $\Delta\mu_{AB} \equiv 2\pi \Delta Q_{AB}$ for simplicity.
For $\Delta Q_{AB} \ll Q_A$ and $\Delta Q_{AB} \ll Q_B$,

$$\begin{aligned} \left(\Delta KL\right)^2 \beta_x\beta_y \sin^2\mu &= \left(\cos 2\pi Q_A - \cos 2\pi Q_B\right)^2 \\ &= \left(\cos\mu\cos\Delta\mu_{AB} - \sin\mu\sin\Delta\mu_{AB} - \cos\mu\cos\Delta\mu_{AB} + \sin\mu\sin\left(-\Delta\mu_{AB}\right)\right)^2 \\ &\simeq \left(\cos\mu - \Delta\mu_{AB}\sin\mu - \cos\mu - \Delta\mu_{AB}\sin\mu\right)^2 \\ &\simeq 4\,\Delta\mu_{AB}^2 \sin^2\mu \\ &\simeq \left(2\pi\right)^2\left(Q_A - Q_B\right)^2 \sin^2\mu \end{aligned}$$

which reproduces Handbook 4.5.4 Eq. 31: $\nu_A - \nu_B \simeq \frac{1}{2\pi}\frac{\sqrt{\beta_x\beta_y}}{f}$.

Obtaining normal and skew quad strengths from tune shifts

The form for the tune shifts due to skew quad errors:

$$\left(\cos 2\pi\left(Q_x + \Delta Q_x\right) - \cos 2\pi\left(Q_y + \Delta Q_y\right)\right)^2 - \left(\cos 2\pi Q_x - \cos 2\pi Q_y\right)^2 = \left(\Delta KL\right)^2 \beta_x\beta_y \sin 2\pi Q_x \sin 2\pi Q_y \quad (1)$$

can be compared to the formula obtained for the normal tune shifts (Wille Eq. 3.272):

$$2\left(\cos 2\pi\left(Q+\Delta Q\right) - \cos 2\pi Q\right) = -\Delta KL\beta\sin 2\pi Q \quad (2)$$

Since a skew quad term makes equal-sign contributions to the transport matrix elements R_{21} and R_{43} (see matrix representation below), the normal quad term can be extracted from the difference of ΔKL values obtained using the horizontal and vertical tune shifts.

The left side of this form simplifies under $\Delta Q_x \ll Q_x$ and $\Delta Q_y \ll Q_y$ to

$$4\pi\ \left(\cos 2\pi Q_x - \cos 2\pi Q_y\right)\left(-\Delta Q_x \sin 2\pi Q_x + \Delta Q_y \sin 2\pi Q_y\right) = \left(\Delta KL\right)^2 \beta_x\beta_y \sin 2\pi Q_x \sin 2\pi Q_y$$

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Matrix representation

Transport matrix for an element with normal quad strength KL and skew quad strength K_sL in the approximations $\sqrt{K}L \ll 1$ and $\sqrt{K_s}L \ll 1$ and length L:

$M_{\mathrm{ab}} M_L M_{\mathrm{ab}} =$

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{KL}{2} & 1 & \frac{K_sL}{2} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{K_sL}{2} & 0 & \frac{-KL}{2} & 1 \end{bmatrix}
\begin{bmatrix} 1 & L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{KL}{2} & 1 & \frac{K_sL}{2} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{K_sL}{2} & 0 & \frac{-KL}{2} & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 + \frac{KL^2}{2} & L & \frac{K_sL^2}{2} & 0 \\ KL + \frac{(K+K_s)^2L^3}{4} & 1 + \frac{KL^2}{2} & K_sL & \frac{K_sL^2}{2} \\ \frac{K_sL^2}{2} & 0 & 1 - \frac{KL^2}{2} & L \\ K_sL & \frac{K_sL^2}{2} & -KL + \frac{(K+K_s)^2L^3}{4} & 1 - \frac{KL^2}{2} \end{bmatrix}
$$

For $K = 0$ this becomes

$$
\begin{bmatrix} 1 & L & \frac{K_sL^2}{2} & 0 \\ \frac{K_s^2L^3}{4} & 1 & K_sL & \frac{K_sL^2}{2} \\ \frac{K_sL^2}{2} & 0 & 1 & L \\ K_sL & \frac{K_sL^2}{2} & \frac{K_s^2L^3}{4} & 1 \end{bmatrix}
$$

The drift length L does not contribute to the skew transport.
The skew quad strength K_sL makes a second-order, same-sign contribution to the H and V tunes.

For $K_s = 0$ this becomes

$$
\begin{bmatrix} 1 + \frac{KL^2}{2} & L & 0 & 0 \\ KL + \frac{K^2L^3}{4} & 1 + \frac{KL^2}{2} & 0 & \\ 0 & 0 & 1 - \frac{KL^2}{2} & L \\ 0 & 0 & -KL + \frac{K^2L^3}{4} & 1 - \frac{KL^2}{2} \end{bmatrix}
$$

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Tune Shifts from Normal and Skew Quad Contributions

Skew Quad

$$\frac{qL}{P_0} B_Y = K_2 L \left(x^2 - y^2\right)$$

$$\frac{qL}{P_0} B_X = 2K_2 L \, x \, y$$

$$b1 = \frac{1}{2!} \frac{qL}{P_0} \frac{dB_Y}{dx} = K_2 L \, x$$

$$a1 = \frac{1}{2!} \frac{qL}{P_0} \frac{dB_X}{dx} = K_2 L \, y$$

Assuming the initial sextupole strength is zero, changes in the sextupole strength $\mathbf{K_2}$ gives changes in the local field slopes (normal (b_1) and skew (a_1) quad strength changes):

$$\Delta b_1 = \frac{qL}{P_0} \Delta \left(\frac{dB_Y}{dx}\right) = 2\Delta K_2 L \left(X_0 + \Delta x\right)$$

$$\Delta a_1 = \frac{qL}{P_0} \Delta \left(\frac{dB_X}{dx}\right) = 2\Delta K_2 L \left(Y_0 + \Delta y\right)$$

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Normal and Skew Quad Contributions from Sextupoles

Tune shifts from skew $\boldsymbol{a_1}$ (Eq. 1)

$$(\cos 2\pi\,(Q_x + \Delta Q_x) - \cos\,(Q_y + \Delta Q_y))^2 - (\cos 2\pi Q_x - \cos 2\pi Q_y)^2 = a_1^2 \beta_x \beta_y \sin 2\pi Q_x \sin 2\pi Q_y \tag{3}$$

Factor $\alpha^2 - \beta^2 = [\alpha - \beta][(\alpha + \beta)]$:

$$[\cos 2\pi\,(Q_x + \Delta Q_x) - \cos\,(Q_y + \Delta Q_y)) - (\cos 2\pi Q_x - \cos 2\pi Q_y]\,[...+...] = a_1^2 \beta_x \beta_y \sin 2\pi Q_x \sin 2\pi Q_y \equiv A^2 \tag{4}$$

Tune shifts from normal $\boldsymbol{b_1}$ (Eq. 2)

$$\cos 2\pi\,\left(Q_{x/y} + \Delta Q_{x/y}\right) - \cos 2\pi Q_{x/y} = -\frac{b_1}{2} \beta_{x/y} \sin 2\pi Q_{x/y} \tag{5}$$

Subtract $\boldsymbol{Y}$ equation from $\boldsymbol{X}$ equation:

$$(\cos 2\pi\,(Q_x + \Delta Q_x) - \cos 2\pi Q_x) - (\cos 2\pi\,(Q_y + \Delta Q_y) - \cos 2\pi Q_y) = -\frac{b_1}{2}\,(\beta_x \sin 2\pi Q_x - \beta_y \sin 2\pi Q_y) \equiv B \tag{6}$$

Substitute Eq. 6 into Eq. 4:

$$B\,[\cos 2\pi\,(Q_x + \Delta Q_x) - \cos\,(Q_y + \Delta Q_y)) + (\cos 2\pi Q_x - \cos 2\pi Q_y] = A^2 \tag{7}$$

Multiply Eq. 6 by $\boldsymbol{B}$ and add to Eq. 7:

$$2\ B\,[\cos 2\pi\,(Q_x + \Delta Q_x) - \cos 2\pi Q_x] = A^2 + B^2 \tag{8}$$

Multiply Eq. 6 by $\boldsymbol{B}$ and subtract from Eq. 7:

$$2\ B\,[\cos 2\pi\,(Q_y + \Delta Q_y) - \cos 2\pi Q_y] = A^2 - B^2 \tag{9}$$

This needs more work. The split in the tunes in the skew case is given by changing the sign of $\boldsymbol{A^2}$.
By symmetry $\cos 2\pi\,(Q_x + \Delta Q_x) - \cos 2\pi Q_x = -\,[\cos 2\pi\,(Q_y + \Delta Q_y) - \cos 2\pi Q_y]$, so

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Tune shifts from skew quad with strength $\boldsymbol{a_1}$ (See talk of 5 January 2022)

$$(\cos(\mu_x + \Delta\mu_x) - \cos(\mu_y + \Delta\mu_y))^2 - (\cos\mu_x - \cos\mu_y)^2 = a_1^2\beta_x\beta_y \sin\mu_x \sin\mu_y$$

After Georg's Mathematica analysis based on Sagan and Rubin *Linear analysis of coupled lattices*, Phys.Rev.S.T. Vol 2, 074001 (1999), for $\Delta\mu_x, \Delta\mu_y \ll 1$,

$$\Delta\mu_x = \frac{-a_1^2\beta_x\beta_y \sin\mu_y}{4(\cos\mu_x - \cos\mu_y)} + O(a_1^4)$$

$$\Delta\mu_y = \frac{a_1^2\beta_x\beta_y \sin\mu_x}{4(\cos\mu_x - \cos\mu_y)} + O(a_1^4)$$

Reminder special case $\mu_x = \mu_y \equiv \mu$, (talk of 5 January 2022 and Handbook 4.5.4 Eq. 31)

$$\Delta\mu \simeq \frac{a_1\sqrt{\beta_x\beta_y}}{2}$$

Linear in $\boldsymbol{a_1}$!

Warning from DCS: there are additional terms (including linear in $\boldsymbol{a_1}$) if the unperturbed lattice is already coupled.

Two approaches:

- Add the coupling matrix to the Mathematica analysis,
- Run simulations in CesrV in the lattice optimized to a phase measurement.