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Wyatt Carbonell (REU) & JAC

Correction to the Beam Size Calculation Due to the Finite Length of the Sextupole

The horizontal angle change due to sextupole strength change integrated over sextupole length is given by

$$\langle \Delta p_{\rm x} \rangle_{\rm L} \;=\; \int_0^{\rm L} {\rm d}s \frac{{\rm d}\Delta p_{\rm x}(s)}{{\rm d}s} \tag{1}$$

where

$$ {\rm d}\Delta p_{\rm x}(s) \;=\; -{\rm d}s \frac{\Delta K_2}{2}\left(x^2(s) + \sigma_{\rm x}^2\right) \tag{2}$$

Here $x(s)$is the distance of the beam from the sextupole center and $\sigma_{\rm x}$ is the horizontal size of the beam.
Writing $x(s)$ as

$$x(s) \;=\; X_0 + p_{\rm x}(s) ds \tag{3}$$

where X_0 is the beam position at the entrance, and

$$p_{\rm x}(s) \;=\; P_{\rm x0} + \Delta p_{\rm x}(s) \tag{4}$$

where $P_{\rm x0}$ is the entrance angle, and using Eq. 2, we obtain

$$\langle \Delta p_{\rm x} \rangle_{\rm L} \;=\; -\frac{\Delta K_2 L}{2}\left[\left(X_0^2 + \sigma_{\rm x}^2\right) + \int_0^{\rm L} {\rm d}s \int_0^{s'} ds' \; 2X_0\left(P_{\rm x0} + \Delta p_{\rm x}(s')\right)\right]. \tag{5}$$

Keeping only terms linear in ΔK_2, we have

$$\langle \Delta p_{\rm x} \rangle_{\rm L} \;=\; -\frac{\Delta K_2 L}{2}\left(X_0^2 + \sigma_{\rm x}^2 + X_0 P_{\rm x0} L\right). \tag{6}$$

Since $P_{\rm x0}$ can reach values of milliradians, this correction term can be of order tenths of mm^2, which is comparable to the measurement error in the beam size. However, as we will see below, it is cancelled by the length correction to the quadrupole term Δb_1.

The change from the prior approximation to the term in the beam size calculation is given by

$$-2\left\langle \frac{\Delta p_{\rm x}}{\Delta K_2 L}\right\rangle_{\rm L} \;=\; -2\frac{\Delta p_{\rm x}}{\Delta K_2 L} + X_0 P_{\rm x0} L \tag{7}$$

Repeating the above exercise for the quadrupole term caused by the sextupole strength change, we find

$$\langle \boldsymbol{\Delta b_1} \rangle_{\mathrm{L}} \;=\; \boldsymbol{\Delta K_2 L} \left(\boldsymbol{X_0} + \frac{\boldsymbol{P_{\mathrm{x}0} L}}{\mathbf{2}} \right). \tag{8}$$

This correction can be of order 0.1 mm, which is comparable to our uncertainty in calculating the beam size. The change from the prior approximation to the term in the beam size calculation is given by

$$\left\langle \frac{\boldsymbol{\Delta b_1}}{\boldsymbol{\Delta K_2 L}} \right\rangle_{\mathrm{L}} \;=\; \frac{\boldsymbol{\Delta b_1}}{\boldsymbol{\Delta K_2 L}} + \frac{\boldsymbol{P_{\mathrm{x}0} L}}{\mathbf{2}} \tag{9}$$

And now for the surprise. We calculate the squared beam size using the length-corrected values:

$$\boldsymbol{\sigma_{\mathrm{x}}^2} \;=\; -\mathbf{2} \left\langle \frac{\boldsymbol{\Delta p_{\mathrm{x}}}}{\boldsymbol{\Delta K_2 L}} \right\rangle_{\mathrm{L}} - \left(\left\langle \frac{\boldsymbol{\Delta b_1}}{\boldsymbol{\Delta K_2 L}} \right\rangle_{\mathrm{L}} \right)^{\mathbf{2}} \tag{10}$$

$$\;=\; -\mathbf{2} \frac{\boldsymbol{\Delta p_{\mathrm{x}}}}{\boldsymbol{\Delta K_2 L}} + \boldsymbol{X_0 P_{\mathrm{x}0} L} - \left[\boldsymbol{X_0^2} + \boldsymbol{X_0 P_{\mathrm{x}0} L} + \left(\frac{\boldsymbol{P_{\mathrm{x}0} L}}{\mathbf{2}} \right)^{\mathbf{2}} \right] \tag{11}$$

The linear terms in the length correction in the beam size calculation cancel exactly!

The extension to both transverse coordinates gives the same result for the length-correction cancellation in the $\boldsymbol{\Delta a_1}$ and $\boldsymbol{\Delta p_{\mathrm{y}}}$ terms as well.