INVESTIGATION OF SYSTEMATIC CONTRIBUTIONS TO AND ERROR IN THE MEASUREMENT OF BEAM SIZE USING SEXTUPOLE MAGNETS

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Abstract

Owing to the quadratic field dependence of sextupole magnets, a variation in the strength of a sextupole results in a beam size dependent orbit kick and change in the betatron tune. By measuring both, we can produce an estimate of the beam size in the sextupole. We present a full derivation of the equations necessary to produce such an estimate, including as of yet unconsidered correction terms relating to the finite length of the sextupole. We analyze data from the Cornell Electron Storage Ring (CESR), and show that these correction terms are small enough to disregard, in general. In addition, we discuss the determination of error in our measurement, the limits of our precision, and report on the current state of the measurement.

INTRODUCTION

We report on the status of a project to measure the beam size in a storage ring using variations in the strength of sextupole magnets, as begun by J. Crittenden and G. Hoffstaetter. The first iteration of this method was proposed in 2021, considering the one dimensional case and proposing methods to measure the beam size from tune and orbit measurements [1]. In 2022, the method was extended to the two transverse dimensions of the beam cross section, and an improved method for determining the value of the entrance beam positions, necessary for the beam size calculation, was developed [2]. The latest report finds that the determination of the entrance beam positions is reproducible by comparing multiple measurement methods, including a reliable method to determine quadrupole kick strengths from the horizontal and vertical tune change, and improves upon the analysis process enough to provide results for all 153 data scans [3].

In storage rings, sextupoles are typically used to correct for higher order effects, especially those of chromatic aberration [4]. However, sextupoles have a unique, quadratic magnetic field dependence on position. This property allows for integration of the Lorentz force over the width of the beam, which leads to an expression for a beam size dependent orbit kick within a sextupole. We take advantage of this property to measure the beam size indirectly. In the following section, we derive a variant of these expressions which account for the finite length of the sextupole. This leaves us with approximate expressions which can be applied to data in Section 2, where we show that the magnitude of the finite length correction is sufficiently small to be disregarded. In Section 3, we detail how error estimates are produced by means of least squares regression. This allows us to make statements about the typical error in our measurements in Section 4. In Section 5, we investigate the relationship between the number and range of measurements we take and the error in the determined beam size. In Section 6, we discuss the status of the measurement method, and identify potential issues in need of further investigation.

DERIVATION OF BEAM SIZE EXPRESSIONS

In this section, we will derive equations necessary for our measurement of the beam size. For consistency with the existing work done on this project, we will ultimately derive expressions consistent with coordinates with origin at the center of the sextupole. However, we find it illustrative to first derive expressions relative to coordinates with origin at the beginning of the sextupole, and to later transform our coordinate system. The equations presented will model those from [2], but with the addition of length dependent terms.

Horizontal Orbit Kick Derivation

We can express the average horizontal orbit kick across the sextupole, $\langle \Delta p_{\mathrm{x}} \rangle_{\mathrm{L}}$, as an integral of infinitesimal contributions $\frac{\mathrm{d}p_{\mathrm{x}}}{\mathrm{d}s}$ over the length L of the sextupole via the relation

$$\langle \Delta p_{\mathrm{x}} \rangle_{\mathrm{L}} = \int_0^L \frac{\mathrm{d}p_{\mathrm{x}}}{\mathrm{d}s} \mathrm{d}s \tag{1}$$

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To find the kick element $\frac{dp_x}{ds}$, we will consider the force on the beam, given by the Lorentz force

$$\vec{F} = q\vec{v} \times \vec{B} \tag{2}$$

We assume the beam charge distribution is approximately Gaussian, centered around the horizontal beam position relative to the center of the sextupole, x, and with a horizontal root mean square (RMS) width σ_x. We note that the quantity x varies as the beam travels, and so is a function of the position s. After appropriate normalization, we find

$$q_x = \frac{q_0}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \frac{-(x'-x)^2}{2\sigma_x^2} dx' \tag{3}$$

where q_0 is related to the total charge of the particle bunch. The magnitude of the magnetic field is given by the relationship for a sextupole,

$$B_y(x) = \frac{1}{2}\frac{d^2 B_y}{dx^2} x^2 \tag{4}$$

In addition, B_y is defined for a sextupole by

$$B_y = \frac{p_0}{q_0}\frac{K_2}{2}(y^2 - x^2) \tag{5}$$

which leads to the second derivative

$$\frac{d^2 B_y}{dx^2} = -\frac{p_0}{q_0} K_2 \tag{6}$$

where K_2 is the strength of the sextupole field and p_0 is the momentum of a particle in the beam. We note that the magnetic field expression used is one such that the horizontal component is positive in the direction of the center of the storage ring. Combining the magnetic field and charge distribution with the Lorentz force, along with the assumptions that the particles are strongly relativistic, and the magnetic field is perfectly perpendicular to the particles' velocity, gives us an expression for the force on the beam averaged over the horizontal direction

$$\begin{aligned}\langle F\rangle_x &= -\frac{q_0 c}{\sigma_x\sqrt{2\pi}}\frac{1}{2}\frac{p_0}{q_0}K_2 \int_{-\infty}^{\infty} (x')^2 \exp \frac{-(x'-x)^2}{2\sigma_x^2} dx' \\ &= -\frac{K_2}{2} p_0 c (x^2 + \sigma_x^2)\end{aligned} \tag{7}$$

The quadratic dependence on position in a sextupole magnet is noteworthy here, providing the necessary factor of $(x')^2$ in the force integral. Likewise, we will consider a vertical cross-section of the beam bunch as a Gaussian charge distribution of RMS width σ_y, centered a vertical distance y from the center of the sextupole,

$$q_y = \frac{q_0}{\sigma_y\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \frac{-(y'-y)^2}{2\sigma_y^2} dy' \tag{8}$$

in combination with a corresponding expression for the magnetic field

$$B_x(y) = \frac{1}{2}\frac{d^2 B_y}{dy^2} y^2 = \frac{1}{2}\frac{p_0}{q_0} K_2 y^2 \tag{9}$$

which, in effect, differs from the horizontal equivalent by a factor of -1. In a clear parallel, the force averaged over the vertical direction is given by

$$\begin{aligned}\langle F\rangle_y &= \frac{q_0 c}{\sigma_y\sqrt{2\pi}}\frac{1}{2}\frac{p_0}{q_0}K_2 \int_{-\infty}^{\infty} (y')^2 \exp \frac{-(y'-y)^2}{2\sigma_y^2} dy' \\ &= \frac{K_2}{2} p_0 c (y^2 + \sigma_y^2)\end{aligned} \tag{10}$$

We can now recombine into a single, averaged force,

$$\langle F\rangle_{x+y} = \langle F\rangle_x + \langle F\rangle_y = \frac{K_2}{2} p_0 c (y^2 + \sigma_y^2 - x^2 - \sigma_x^2) \tag{11}$$

We consider a small change in the sextupole strength ΔK_2, which causes a change in the position of the beam by an amount Δx in the horizontal direction, and a change of Δy in the vertical direction. Working in units of $p_0 c = 1$, taking an initial value $K_2 = 0$, and disregarding terms non-linear in ΔK_2, we can write the infinitesimal horizontal orbit kick as the difference in the force before and after the change,

$$\frac{dp_x}{ds} = \frac{\Delta K_2}{2}(y^2(s) - x^2(s) + \sigma_y^2 - \sigma_x^2) \tag{12}$$

where we have now explicitly expressed x and y as functions of the orbital position s. This allows us to write an expression for the horizontal orbit kick averaged over the finite length, L, of the sextupole,

$$\begin{aligned}\langle \Delta p_x\rangle_L &= \int_0^L \frac{dp_x}{ds} ds \\ &= \int_0^L \frac{\Delta K_2}{2}(y^2(s) - x^2(s) + \sigma_y^2 - \sigma_x^2) ds\end{aligned} \tag{13}$$

If we consider this integral as a term in x and a term in y,

$$\langle \Delta p_{\mathrm{x}} \rangle_{\mathrm{L}} = \int_0^L \frac{\Delta K_2}{2}(y^2(s) + \sigma_{\mathrm{y}}^2)\mathrm{d}s - \int_0^L \frac{\Delta K_2}{2}(x^2(s) + \sigma_{\mathrm{x}}^2)\mathrm{d}s \tag{14}$$

then we can make use of a symmetrical argument to quickly evaluate one expression after knowing the other. Focusing on the horizontal integral component, we can express the horizontal beam position $x(s)$, now explicitly a function of position, as

$$x(s) = X_0 + \Delta x(s) \tag{15}$$

where $\Delta x(s)$ is the change in horizontal position within the sextupole. This change in position is related to the beam angle, $p_{\mathrm{x}}(s')$, by

$$\Delta x(s) = \int_0^s p_{\mathrm{x}}(s')\mathrm{d}s' \tag{16}$$

We expand $x^2(s)$ and rearrange the resulting equation into two terms,

$$\begin{aligned}\langle \Delta p_{\mathrm{x}} \rangle_{\mathrm{L}} &= -\int_0^L \frac{\Delta K_2}{2}(X_0^2 + 2X_0\Delta x(s) + \Delta x^2(s) + \sigma_{\mathrm{x}}^2)\mathrm{d}s \qquad (17)\\ &= -\int_0^L \frac{\Delta K_2}{2}(X_0^2 + \sigma_{\mathrm{x}}^2)\mathrm{d}s \\ &\quad -\int_0^L \frac{\Delta K_2}{2}(2X_0\Delta x(s) + \Delta x^2(s))\mathrm{d}s\end{aligned}$$

The first term represents the kick from a sextupole of zero length, in which $x(s) = X_0$, and is easily evaluated, giving us

$$-\int_0^L \frac{\Delta K_2}{2}(X_0^2 + \sigma_{\mathrm{x}}^2)\mathrm{d}s = -\frac{\Delta K_2 L}{2}(X_0^2 + \sigma_{\mathrm{x}}^2) \tag{18}$$

We are now tasked with finding an expression for $\Delta x(s)$, so that the correction term can be computed. Similarly to the treatment of $x(s)$, we will express $p_{\mathrm{x}}(s')$ as

$$p_{\mathrm{x}}(s') = P_{\mathrm{X0}} + \Delta p_{\mathrm{x}}(s') \tag{19}$$

where P_{X0} is the entrance angle and $\Delta p_{\mathrm{x}}(s')$ is the cumulative bend angle caused by the sextupole up to the point s'. The factor $\Delta p_{\mathrm{x}}(s')$ is small compared to P_{X0}. Thus, we approximate the beam angle as constant within the sextupole, and equal to the entrance angle,

$$p_{\mathrm{x}}(s') = P_{\mathrm{X0}} \tag{20}$$

We are now able to easily calculate $\Delta x(s)$, which evaluates to

$$\Delta x(s) = P_{\mathrm{X0}} s \tag{21}$$

The leading correction term is given by

$$\begin{aligned}-\int_0^L \Delta K_2 X_0 \Delta x(s)\mathrm{d}s &= -\int_0^L \Delta K_2 X_0 P_{\mathrm{X0}} s\mathrm{d}s \qquad (22)\\ &= -\frac{\Delta K_2 X_0 P_{\mathrm{X0}}}{2} s^2\Big|_{s=0}^{s=L} \\ &= -\frac{\Delta K_2 X_0 P_{\mathrm{X0}}}{2} L^2\end{aligned}$$

The second order correction can also be calculated simply using the tools we have developed, giving

$$\begin{aligned}-\int_0^L \frac{\Delta K_2}{2}\Delta x^2(s)\mathrm{d}s &= -\int_0^L \frac{\Delta K_2}{2} P_{\mathrm{X0}}^2 s^2\mathrm{d}s \qquad (23)\\ &= -\frac{\Delta K_2 P_{\mathrm{X0}}^2}{6} s^3\Big|_{s=0}^{s=L} \\ &= -\frac{\Delta K_2 P_{\mathrm{X0}}^2}{6} L^3\end{aligned}$$

Thus, we have an expression for the horizontal component of Equation (14),

$$\begin{aligned}&-\int_0^L \frac{\Delta K_2}{2}(x^2(s) + \sigma_{\mathrm{x}}^2)\mathrm{d}s \qquad (24)\\ &= -\frac{\Delta K_2 L}{2}(X_0^2 + \sigma_{\mathrm{x}}^2 + X_0 P_{\mathrm{X0}} L - \frac{1}{3}P_{\mathrm{X0}}^2 L^2)\end{aligned}$$

An exactly analogous process gives the vertical component. Thus, we can finally write

$$\begin{aligned}\langle \Delta p_{\mathrm{x}} \rangle_{\mathrm{L}} &= \frac{\Delta K_2 L}{2}(Y_0^2 - X_0^2 + \sigma_{\mathrm{y}}^2 - \sigma_{\mathrm{x}}^2 \qquad (25)\\ &\quad + Y_0 P_{\mathrm{Y0}} L - X_0 P_{\mathrm{X0}} L + \frac{1}{3}P_{\mathrm{Y0}}^2 L^2 - \frac{1}{3}P_{\mathrm{X0}}^2 L^2)\end{aligned}$$

Vertical Orbit Kick Derivation

For completeness, we will derive a corresponding expression for the vertical orbit kick. We will once again integrate over infinitesimal contributions,

$$\left\langle \Delta p_{\mathrm{y}} \right\rangle_{\mathrm{L}} = \int_0^L \frac{\mathrm{d}p_{\mathrm{y}}}{\mathrm{d}s}\mathrm{d}s \tag{26}$$

We find the infinitesimal contribution, $\frac{\mathrm{d}p_\mathrm{y}}{\mathrm{d}s}$, by starting with the relation

$$p_\mathrm{y} = K_2 L X_0 Y_0 \tag{27}$$

We consider a small variation in the strength of the sextupole, ΔK_2, which causes a change in both the horizontal beam position, Δx, and the vertical beam position, Δy. Taking the difference before and after the variation in strength, assuming $K_2 = 0$ initially,

$$\Delta p_\mathrm{y} = \Delta K_2 L x(s) y(s) \tag{28}$$

If we consider a sextupole of infinitesimal length ds, we will find an infinitesimal orbit kick,

$$\frac{\mathrm{d}p_\mathrm{y}}{\mathrm{d}s} = \Delta K_2 x(s) y(s) \tag{29}$$

If we use our previous expressions for $x(s)$, and the corresponding expression for $y(s)$, we can integrate to find

$$\begin{aligned}\left\langle \Delta p_\mathrm{y} \right\rangle_\mathrm{L} &= \int_0^L \frac{\mathrm{d}p_\mathrm{y}}{\mathrm{d}s}\mathrm{d}s \\ &= \int_0^L \Delta K_2 x(s) y(s) \mathrm{d}s \\ &= \int_0^L \Delta K_2 (X_0 Y_0 + P_{\mathrm{X}0} Y_0 s + X_0 P_{\mathrm{Y}0} s + P_{\mathrm{X}0} P_{\mathrm{Y}0} s^2)\mathrm{d}s \\ &= \Delta K_2 (X_0 Y_0 L + P_{\mathrm{X}0} Y_0 \frac{L^2}{2} + X_0 P_{\mathrm{Y}0} \frac{L^2}{2} + P_{\mathrm{X}0} P_{\mathrm{Y}0} \frac{L^3}{3})\end{aligned} \tag{30}$$

which we will rearrange into our final expression,

$$\left\langle \Delta p_\mathrm{y} \right\rangle_\mathrm{L} = \Delta K_2 L (X_0 Y_0 + \frac{Y_0 P_{\mathrm{X}0} L + X_0 P_{\mathrm{Y}0} L}{2} + \frac{P_{\mathrm{X}0} P_{\mathrm{Y}0} L^2}{3}) \tag{31}$$

Normal And Skew Quadrupole Kick Derivations

Our derived expressions are capable of estimating the beam size, but require reliable determinations of X_0 and Y_0. To remedy this, we will derive expressions relating the normal and skew quadrupole kicks, Δb_1 and Δa_1, to X_0 and Y_0, respectively.

Beginning with the normal quadrupole kick, Δb_1, we start at the relation

$$b_1 = K_2 L X_0 \tag{32}$$

where K_2 is the sextupole strength, L is the length of the sextupole, and X_0 is the horizontal entrance beam position relative to the center of the sextupole. We consider changing the sextupole strength by an amount ΔK_2, which will also introduce a change in the horizontal distance of the beam from the center of the sextupole by an amount Δx. Taking the difference before and after this change gives us

$$\Delta b_1 = (K_2 + \Delta K_2) L (X_0 + \Delta x) - K_2 L X_0 \tag{33}$$

Assuming that $K_2 = 0$ initially,

$$\Delta b_1 = \Delta K_2 L x(s) \tag{34}$$

where we have taken $x(s) = X_0 + \Delta x$ as before. Treating this as the expression for a zero length sextupole, we can write the infinitesimal contribution to the overall orbit kick as

$$\frac{\mathrm{d}b_1}{\mathrm{d}s} = \Delta K_2 x(s) \tag{35}$$

We calculate the average beam kick over a finite sextupole of length L by integrating over the infinitesimal contributions, which gives

$$\langle \Delta b_1 \rangle_\mathrm{L} = \int_0^L \frac{\mathrm{d}b_1}{\mathrm{d}s}\mathrm{d}s = \int_0^L \Delta K_2 x(s) \mathrm{d}s \tag{36}$$

To evaluate this integral, we will make the exact same argument for the expression of $x(s)$ as before. It follows directly from this point that

$$\langle \Delta b_1 \rangle_\mathrm{L} = \Delta K_2 \int_0^L (X_0 + P_{\mathrm{X}0} s)\mathrm{d}s = \Delta K_2 (X_0 L + P_{\mathrm{X}0} \frac{L^2}{2}) \tag{37}$$

In general, we will use this expression as a means to find X_0, so write

$$X_0 = \frac{\langle \Delta b_1 \rangle_\mathrm{L}}{\Delta K_2 L} - \frac{P_{\mathrm{X}0} L}{2} \tag{38}$$

We can make an exactly parallel argument for the skew quadrupole kick Δa_1 by taking the definition

$$a_1 = K_2 L Y_0 \tag{39}$$

where y is a vertical beam position analogous to x. Following the same steps as above, we arrive at the expression

$$Y_0 = \frac{\langle \Delta a_1 \rangle_\mathrm{L}}{\Delta K_2 L} - \frac{P_{\mathrm{Y}0} L}{2} \tag{40}$$

Coordinate Shift And Beam Size Expressions

Of course, the value of these expressions is their ability to provide a measurement of the beam size, which we can express explicitly by rearranging the horizontal orbit kick expression,

$$\sigma_x^2 = Y_0^2 - X_0^2 - \frac{2\langle\Delta p_x\rangle_L}{\Delta K_2 L} + Y_0 P_{Y0} L - X_0 P_{X0} L + \frac{P_{Y0}^2 L^2}{3} - \frac{P_{X0}^2 L^2}{3} \quad (41)$$

where we have disregarded the term σ_y^2, which is on the order of four hundred times smaller than σ_x^2. Equation (41) is an expression for the beam size using quantities measured from the beginning of the sextupole, but we can simplify considerably by shifting the coordinates of some quantities to the center of the sextupole. We note that $\langle\Delta p_x\rangle_L$ is an integrated effect, and so is coordinate independent. However, had we developed the quadrupole kick expressions in the center of the sextupole, we would have instead written

$$\langle\Delta b_1\rangle_L = \Delta K_2 \int_{-\frac{L}{2}}^{\frac{L}{2}} (X_0 + \int_{-\frac{s}{2}}^{\frac{s}{2}} P_{X0} ds')ds = \Delta K_2 X_0 L \quad (42)$$

This suggests that $\frac{\langle\Delta b_1\rangle_L}{\Delta K_2 L}$ fills the role of X_0 in the center of the sextupole, and likewise for $\frac{\langle\Delta a_1\rangle_L}{\Delta K_2 L}$ and Y_0. We note that the difference in the derivation lies only in the choice of the bounds of integration. This differs from Equation (38) by a factor of $\frac{P_{X0}L}{2}$, suggesting that linear correction terms are a result of the choice of coordinates. To effectively change coordinates to the center of the sextupole, we combine Equation (41) with Equations (38) and (40). We find

$$\sigma_x^2 = (\frac{\langle\Delta a_1\rangle_L}{\Delta K_2 L})^2 - (\frac{\langle\Delta b_1\rangle_L}{\Delta K_2 L})^2 - \frac{2\langle\Delta p_x\rangle_L}{\Delta K_2 L} + \frac{7}{12} L^2 (P_{Y0}^2 - P_{X0}^2) \quad (43)$$

A similar process applied to Equation (31) gives

$$\frac{\langle\Delta p_y\rangle_L}{\Delta K_2 L} = (\frac{\langle\Delta a_1\rangle_L}{\Delta K_2 L})(\frac{\langle\Delta b_1\rangle_L}{\Delta K_2 L}) + \frac{1}{12} P_{X0} P_{Y0} L^2 \quad (44)$$

No first order terms remain in either expression, and only second order correction terms remain.

Beam Size Error Expression Derivation

We note that the following two expressions hold true in general:

$$\delta(\sigma_x^2) = \sqrt{\delta^2(\sigma_x^2)} \quad (45)$$

$$\delta(\sigma_x^2) = 2\sigma_x \delta(\sigma_x) \quad (46)$$

Combining these two expressions gives us an expression for the error in the beam width measurement,

$$\delta(\sigma_x) = \frac{1}{2\sigma_x}\sqrt{\delta^2(\sigma_x^2)} \quad (47)$$

Furthermore, we can express the squared error in the squared beam width, $\delta^2(\sigma_x^2)$, as a sum of squared errors,

$$\delta^2(\sigma_x^2) = (\delta(Y_0^2))^2 + (-\delta(X_0^2))^2 + (-2\delta(\frac{\langle\Delta p_x\rangle_L}{\Delta K_2 L}))^2 + (\frac{7}{36}\delta(P_{Y0}^2 L^2))^2 + (-\frac{7}{36}\delta(P_{X0}^2 L^2))^2 \quad (48)$$

We will express each term in a way that is more convenient for applications to measurements by using

$$\delta(Y_0^2) = 2Y_0\delta(Y_0) \quad (49)$$

$$\delta(X_0^2) = 2X_0\delta(X_0) \quad (50)$$

$$\delta(P_{Y0}^2 L^2) = 2P_{Y0} L^2 \delta(P_{Y0}) \quad (51)$$

$$\delta(P_{X0}^2 L^2) = 2P_{X0} L^2 \delta(P_{X0}) \quad (52)$$

This means that the error in the beam width can be expressed as

$$\delta(\sigma_x) = \frac{1}{2\sigma_x}\sqrt{\delta^2(\sigma_x^2)} \quad (53)$$

$$\delta^2(\sigma_x^2) = 4Y_0^2\delta^2(Y_0) + 4X_0^2\delta^2(X_0) + 4\delta^2(\frac{\langle\Delta p_x\rangle_L}{\Delta K_2 L}) + \frac{49}{324} P_{Y0}^2 L^4 \delta^2(P_{Y0}) + \frac{49}{324} P_{X0}^2 L^4 \delta^2(P_{X0}) \quad (54)$$

APPLICATIONS TO MEASUREMENTS

In this section, we will present the application of our derived expressions, including the finite length correction, to data from CESR. This data was collected for all 76 sextupoles in the storage ring, across 153 data scans. We will ultimately conclude that the finite length correction is small enough to disregard in general. For more information regarding the determination of parameters used to calculate the beam size, see References [1-3]. For a detailed description of the determination of the error in our measurements, see Section 3. It is worth noting that the finite length correction changes our beam size measurement on the order of microns, which indicates immediately that it is small enough to disregard. However, we investigate this notion more formally before choosing to disregard the correction.

Figure 1 shows the ratio between the magnitude of the correction to the beam width, σ_{x} to the error in σ_{x}, without correction, for each data scan. The magnitude of the correction to the beam width was calculated via a difference between σ_{x} as calculated directly from Equation (44), and σ_{x} as calculated without the correction term $\frac{7}{12}L^2(P_{\mathrm{Y0}}^2 - P_{\mathrm{X0}}^2)$. The red line marks equality between the correction and the error. We note that the correction is smaller than the error in all cases, as we would expect for a small, second order correction. This suggests that, in general, the length correction can be safely disregarded for the beam size calculation.

Figure 2 shows the percent difference in error in σ_{x}, before and after the second order correction $\frac{7}{12}L^2(P_{\mathrm{Y0}}^2 - P_{\mathrm{X0}}^2)$ is applied to σ_{x}^2, for each data scan. The largest correction to the error is approximately 1.5%, confirming that the correction to the error estimate is also small enough to be negligible.

Since the finite length correction represents a minor correction to our measurements of both the beam size and its error, we will choose to disregard this correction in our future calculations. This amounts to confirming that the thin lens approximation, in which the sextupole field acts only at the center, is sufficiently accurate for our measurement.

DETERMINATION OF ERROR BARS

In this section, we will detail the process by which we determine the magnitude of the error in our measurement. This is done primarily by making use of least squares regression, performed by the FORTRAN library Minuit [5].

All of the errors we determine are, more formally, the value of one standard deviation. Minuit determines the value of one standard deviation in polynomial fit coefficients by

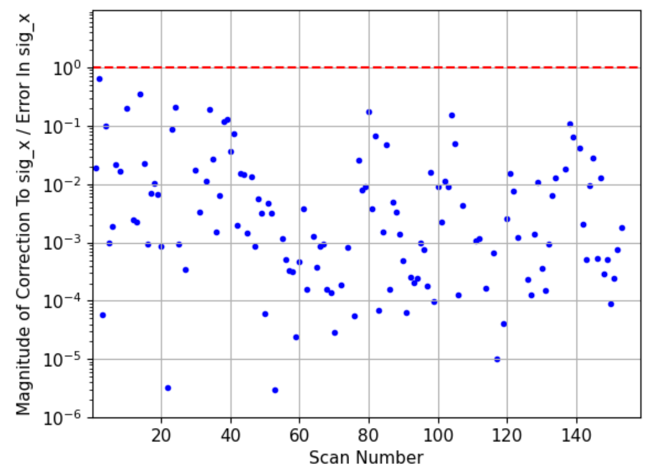

Figure 1: The ratio between the magnitude of the correction to the beam width, σ_{x} to the error in σ_{x}, without correction, for each data scan. The red line marks equality between the correction and the error. We note that the correction is smaller than the error in all cases, as we would expect for the derived, second order correction.

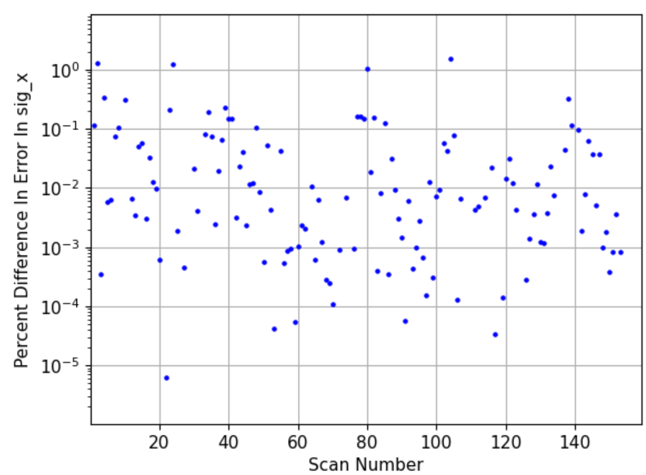

Figure 2: The percent difference in error in σ_{x}, before and after the second order correction $\frac{7}{12}L^2(P_{\mathrm{Y0}}^2 - P_{\mathrm{X0}}^2)$ is applied to σ_{x}^2 for each data scan. The largest correction to the error is approximately 1.5%, confirming that the correction to the error estimate is also negligible.

default [6]. To find the errors in the data, we assume that the error is determined by a known and equivalent source, such as inherent uncertainties in the beam position monitors. This allows for the expression of the errors, as standard deviations, in terms of the variance. If we force the reduced chi squared, the chi squared over the number of degrees of freedom, to be unity, then we can produce a single error estimate for all of the input data. This comes at the expense of any information the reduced chi squared may have provided regarding the quality of the fit [7]. Instead, we determine the quality of the fit by ensuring that the deviations from the fit are on the order of the errors in the input data.

We guarantee that the reduced chi squared is unity via the following process. First, a reasonable estimate for the error in a given quantity is produced by looking at variations in the measured values. This reasonable estimate is used to calculate chi squared, which more than likely differs from the number of degrees of freedom by some factor. By adjusting the error estimate by the square root of this factor, we adjust chi squared such that the reduced chi squared is unity. This fact is verified by once again computing chi squared, and our error estimate is determined.

More specifically, we use the MINOS minimizer, which produces asymmetrical error estimates as defined by a function change [5][6]. A fit which does not include any non-linear effects will, in general, produce a minimum which is quadratic, and thus symmetrical. The difference between the negative and positive error estimates is thus a measure of the non linearity of the fit [5]. While we have not yet investigated the relationship between the negative and positive errors in all cases, we have noted for many cases that they are very similar in magnitude, suggesting that there is not a large non-linear effect impacting our fitted values. The errors we report are the average of the positive and negative error values.

In general, we fit quantities as functions of $\Delta K_2 L$ when determining the beam size. This process results in error estimates for both the individual points used in the fit, and the coefficients on the powers of $\Delta K_2 L$. Figure 3 shows one example of the fitting procedure as applied to Δp_x over $\Delta K_2 L$ in scan 85, on sextupole 10. We note that the reduced chi squared has been forced to unity by adjusting the error estimate. We also note that the variation from the fit is on the order of the error bars, suggesting that the fit was successful. Only the linear coefficient is used in the beam size calculation, as noted in the displayed equation. Δp_x is clearly cubic in $\Delta K_2 L$, and this fact has been lost in our derivation by disregarding terms nonlinear in $\Delta K_2 L$. A full derivation, including nonlinear terms, does predict a cubic relationship.

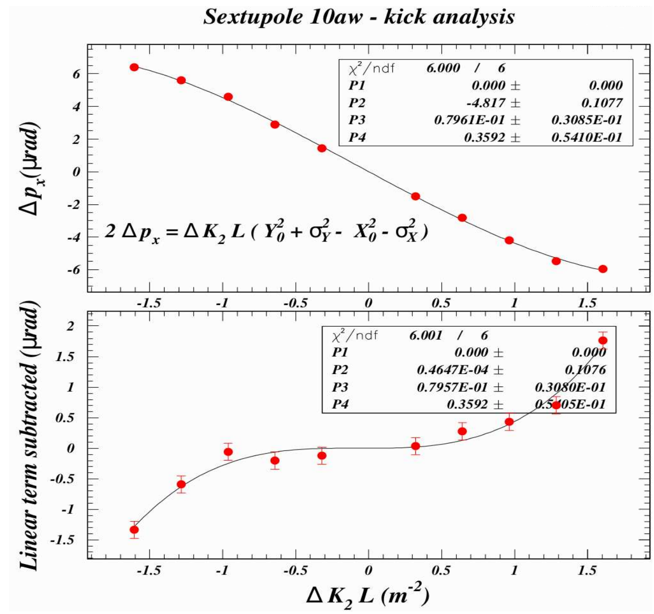

Figure 3: One example of the fitting procedure as applied to the horizontal orbit kick Δp_x over change in sextupole strength $\Delta K_2 L$ in scan 85. We note that the reduced chi squared has successfully been forced to unity by adjusting the error estimate. We also note that the residuals are on the order of the error bars, again suggesting that the fit was successful.

TYPICAL ERROR MAGNITUDES

In this section, we present the current error estimates for the orbit kicks Δp_x and Δp_y, the quadrupole kicks Δb_1 and Δa_1, and the beam size σ_x. We also present typical error values for each quantity. All of the values presented were determined via the method discussed in the previous section.

Figures 4-8 show the errors in the determination of the horizontal orbit kick Δp_x, the vertical orbit kick Δp_y, the normal quadrupole kick Δb_1, the skew quadrupole kick Δa_1, and the beam size σ_x, respectively, over all scans. Each also includes a histogram of the errors, excluding one outlier in Δp_x, seven outliers in Δp_y, one outlier in Δb_1, nine outliers in Δa_1, and eight outliers in σ_x. Their mean values are 0.195 μrad, 0.182 μrad, 28.1 μm^{-1}, 22.5 μm^{-1}, and 0.083 mm, respectively. The root mean square deviations from the mean are 0.157 μrad, 0.177 μrad, 20.2 μm^{-1}, 22.1 μm^{-1}, and 0.055 mm, respectively.

We note that we have succeeded in taking measurements of the beam size with an accuracy close to 0.1 mm. This corresponds to about 10% of the expected beam size measurement, which is on the order of 1 mm. To do so, we need to know the orbit kicks to precision better than microradians and the entrance beam positions to the precision of tens of micrometers. This is difficult to achieve due to the limited range of $\Delta K_2 L$, which, if too large, will cause the beam to be lost in the storage ring. It also means that any systematic effects, even if small, have the potential to substantially impact our measurement, and so need to be considered.

Errors In Δp_x

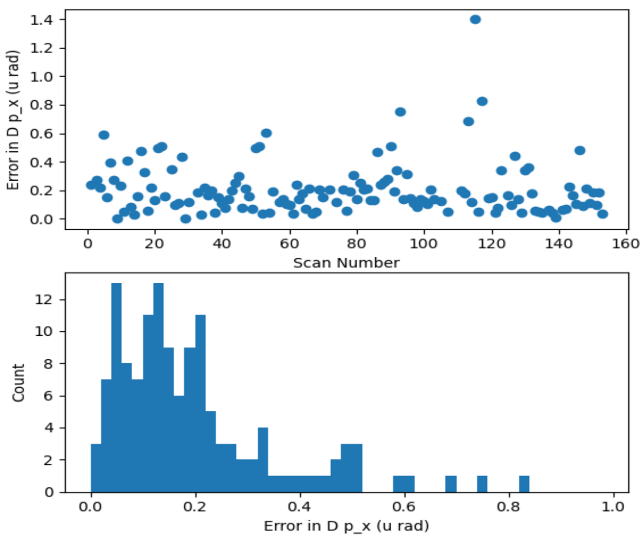

Figure 4: The upper plots shows the error in Δp_x over all scans. The lower plots shows a histogram of these error values, excluding one outlier. The mean value, excluding the outlier, is 0.195 μrad.

Errors In Δp_y

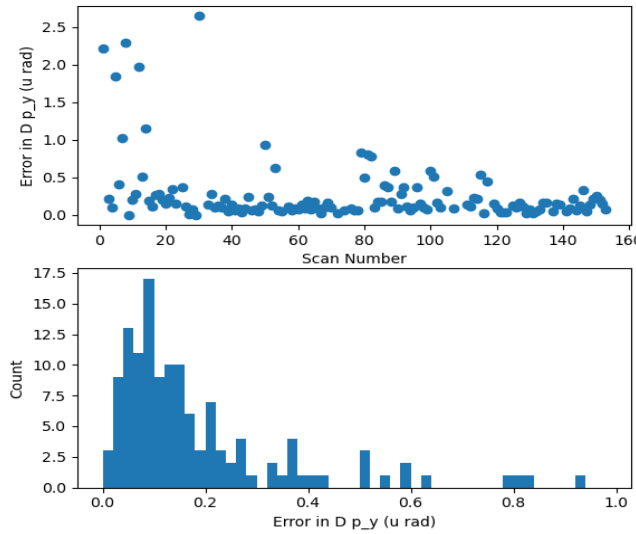

Figure 5: The upper plots shows the error in Δp_y over all scans. The lower plots shows a histogram of these error values, excluding seven outliers. The mean value, excluding the outliers, is 0.182 μrad.

Errors In Δb_1

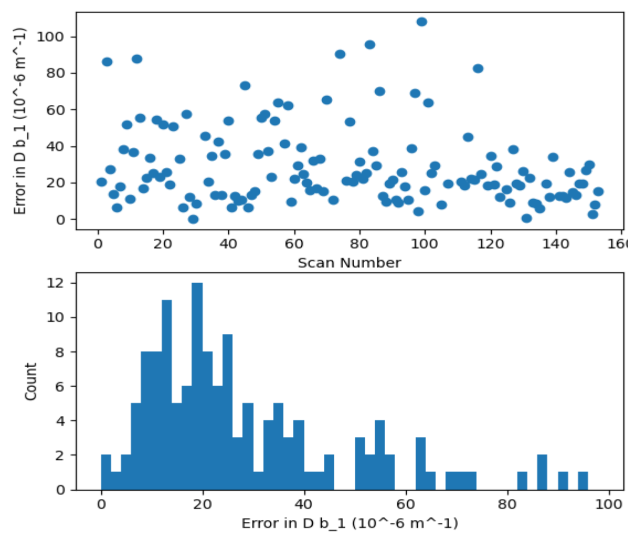

Figure 6: The upper plots shows the error in Δb_1 over all scans. The lower plots shows a histogram of these error values, excluding one outlier. The mean value, excluding the outlier, is 28.1 μm^{-1}.

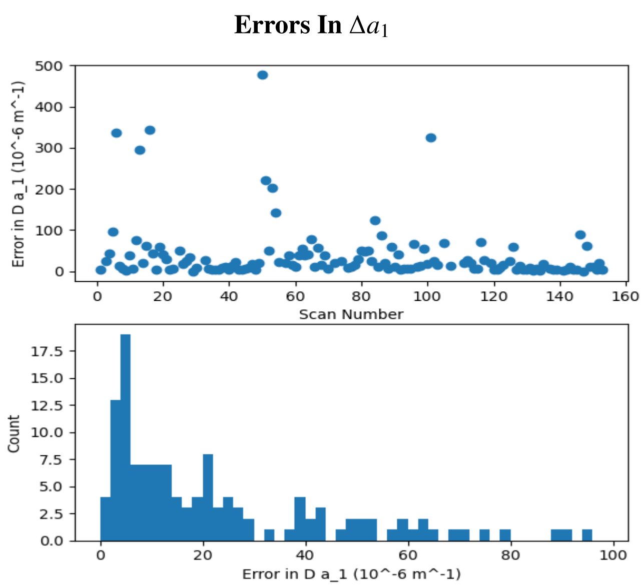

Figure 7: The upper plots shows the error in Δa_1 over all scans. The lower plots shows a histogram of these error values, excluding nine outliers. The mean value, excluding the outliers, is 22.5 μm^{-1}.

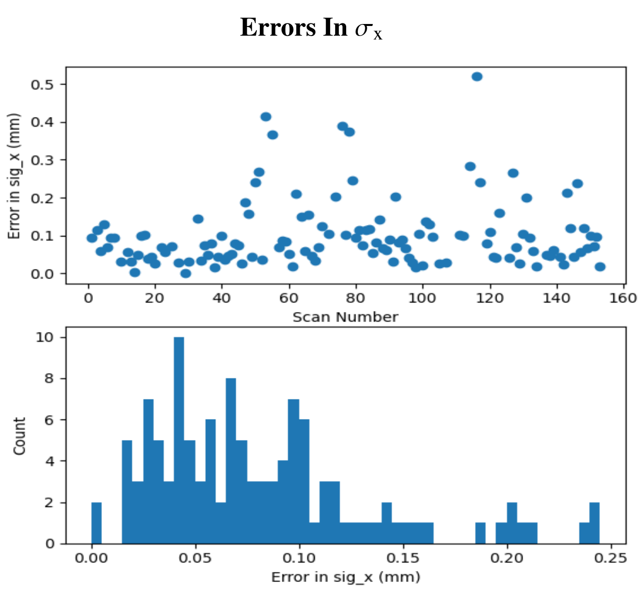

Figure 8: The upper plots shows the error in σ_x over all scans. The lower plots shows a histogram of these error values, excluding eight outliers. The mean value, excluding the outliers, is 0.083 mm.

STATISTICS OF THE ERRORS

In this section, we investigate the relationship between our beam size error estimate and the range of data collected. In particular, we investigate the effect of the range of K_2 values, number of unique K_2 values, and number of points taken at each K_2 value on the error in Δp_x, the largest contributor to the error in the beam size.

Figure 9 shows the error in Δp_x for each of six scans conducted on sextupole number 8 as a function of the range of ΔK_2 values, number of unique ΔK_2 values, and number of points taken at each ΔK_2 value. Sextupole 8 was chosen for this purpose because there happen to be the most scans on sextupole 8 out of all 76 sextupoles. The expected behavior, statistically, is that the error in the measurement decreases and the amount of data increases. We note that scan 113 is an obvious outlier in this plot; we believe that this is because the fit for Δp_x in scan 113 has strongly correlated linear and cubic terms, both of which have errors comparable to their value. Nonetheless, we do not observe a square root dependence of the error on any of these three quantities, unless taking only the most favorable points. We are led to believe that taking more data in an individual scan will not improve the precision of our beam size measurement. This may be due to systematic effects, such as the maximum precision of the beam position monitors in the storage ring.

Conversely, Figure 10 shows the error in Δp_y for each of six scans conducted on sextupole number 8 as a function of the range of ΔK_2 values, number of unique ΔK_2 values, and number of points taken at each ΔK_2 value. We do not observe a square root decrease in the error. Instead, we see an increase in the error as a larger range or larger number of ΔK_2 settings are used. We do see a decrease in the error as the number of points per K_2 value is increased, but it seems to be slower than the expected square root relationship, and highly variable.

The combination of these two facts leads us to believe that six scans is not a large enough sample size to make definitive statements about the statistical dependence of our error estimates on these metrics of the amount of data taken. The safest assumption about this statistical dependence seems to be that the error estimates and amount of data taken are uncorrelated. This would suggest that taking data at more K_2 settings and with more repetition is not worthwhile, at least for this sextupole. A more complete analysis should be conducted, ideally with more data, to determine how many measurements are necessary to produce reliable results.

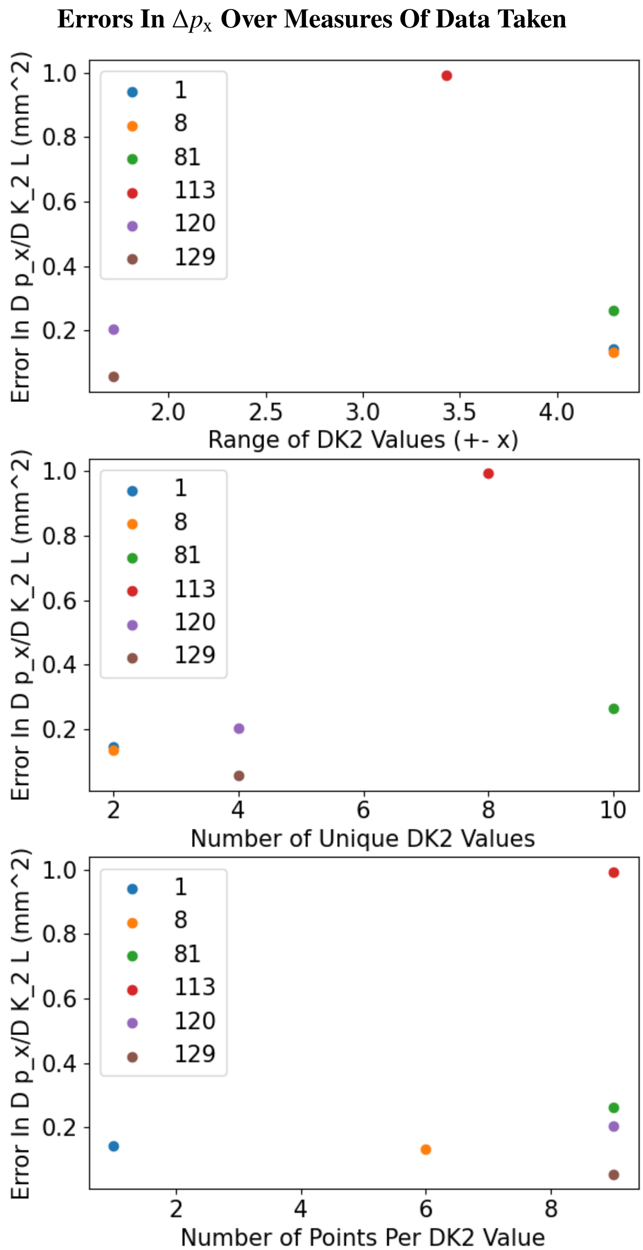

Figure 9: The error in Δp_x for each of six scans conducted on sextupole number 8 as a function of the range of ΔK_2 values, number of unique ΔK_2 values, and number of points taken at each ΔK_2 value. The expected behavior, statistically, is that the error in the measurement decreases and the amount of data increases. We do not observe the expected square root dependence of the error on any of these three quantities, unless taking only the most favorable points. We are led to believe that taking more data in an individual scan will not improve the precision of our beam size measurement.

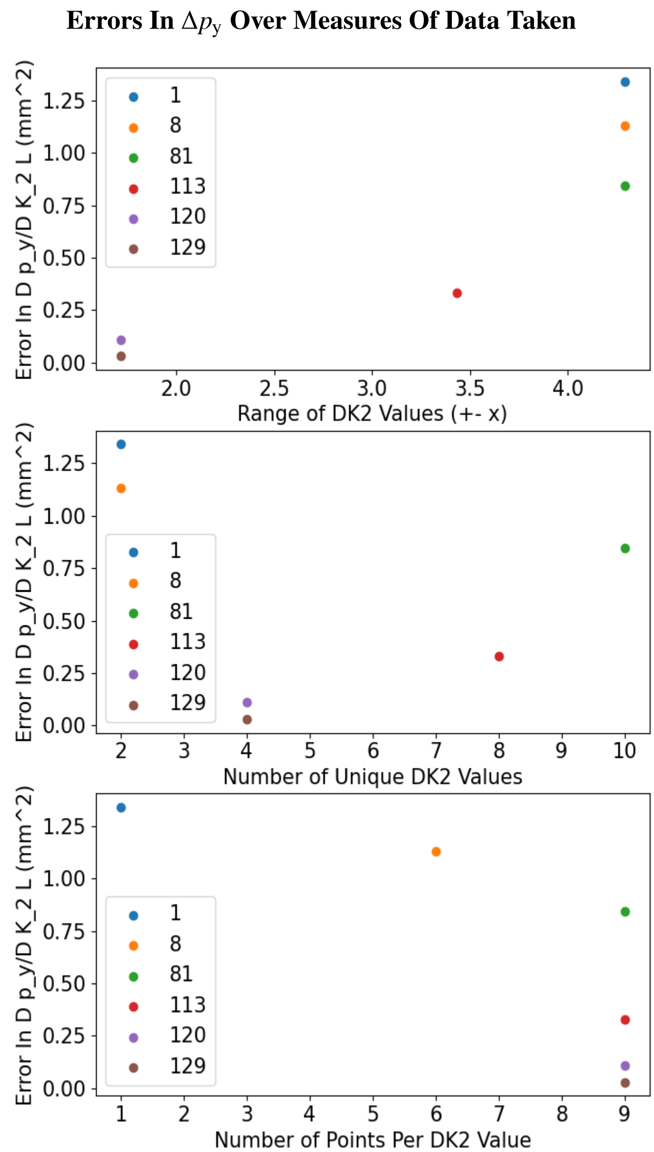

Figure 10: The error in Δp_y for each of six scans conducted on sextupole number 8 as a function of the range of ΔK_2 values, number of unique ΔK_2 values, and number of points taken at each ΔK_2 value. The expected behavior, statistically, is that the error in the measurement decreases and the amount of data increases. We do not observe a square root decrease in the error. Instead, we see an increase in the error as a larger range or larger number of ΔK_2 settings are used. We do see a decrease in the error as the number of points per K_2 value is increased, but it seems to be slower than the expected square root relationship, and highly variable.

IDENTIFYING A SYSTEMATIC ISSUE

In this section, we identify a systematic problem in our beam size measurement. Figure 11 shows the current beam size measurement over all scans in the upper plot, excluding 11 scans in which the current method calculates a negative squared beam size. The lower plot shows the expected beam size from the optics for each scan. Our measured values are overestimated in most, but not all cases. Scans in which the calculated beam size is close to the expected beam size may provide clues as to why most scans fail, and will need to be investigated further.

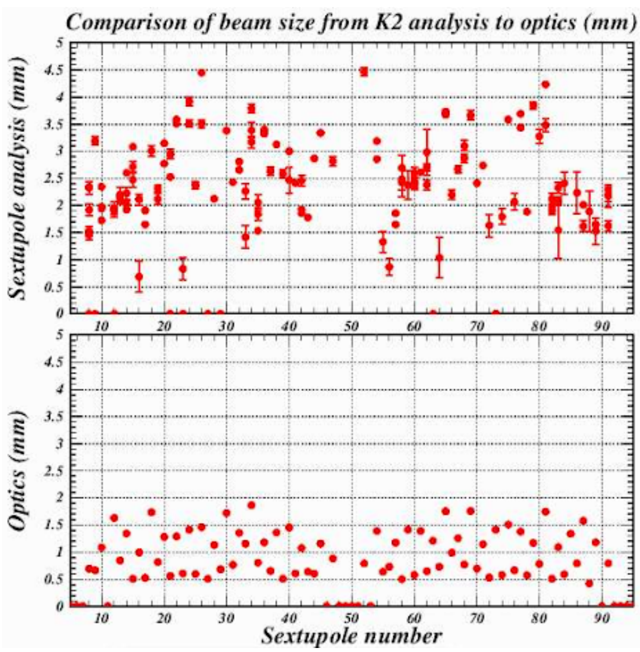

Figure 11: The current beam size measurement over all scans in the upper plot, excluding 11 scans in which the current method calculates a negative squared beam size. The lower plot shows the expected beam size from the optics for each scan. Our measured values are overestimated in most, but not all cases.

One possible explanation for this overestimation is related to our calculated versus measured values of Δp_y. Figure 12 shows the calculated versus measured values of Δp_y over values of $\Delta K_2 L$ for scan 85. We calculate Δp_y according to Equation (31) without length correction, $\Delta p_y = \Delta K_2 L X_0 Y_0$, which applies to the linear term in Δp_y. Scan 85 was chosen for this graph because of its clear linear behavior. However, the slope of our calculated values of Δp_y do not agree with the slope of the measured values, and in fact the calculated slope is significantly too positive.

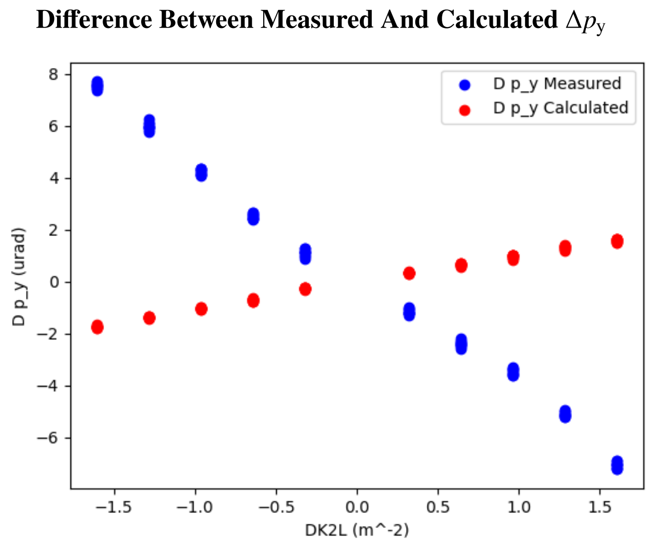

Figure 12: The calculated versus measured values of Δp_y over values of $\Delta K_2 L$ for scan 85. We calculate Δp_y according to Equation (31) without length correction, $\Delta p_y = \Delta K_2 L X_0 Y_0$, which applies to the linear term in Δp_y. The slope of our calculated values of Δp_y do not agree with the slope of the measured values, and in fact the calculated slope is significantly too positive.

Suppose there is some additional field that has impacted our measurements, which we have not accounted for in our theory. These effects are extraneous to the measurement, which applies to orbit kicks and tune changes caused by the sextupole itself. We can estimate the strength of this field by comparing the observed and predicted slopes. In this case, our measured values could be explained by an additional field strength of approximately 22 Gauss, compared to a total change in K_2 of approximately 12 Gauss.

Figure 13 shows measured values of the slopes $\frac{\Delta p_y}{\Delta K_2 L}$ and $\frac{\Delta p_x}{\Delta K_2 L}$ as functions of $\Delta K_2 L$ in scan 85. We would expect these measurements to be approximately constant in $\Delta K_2 L$, but they both show a clear dependence on $\Delta K_2 L$. The fact that these two values fluctuate similarly may suggest a relationship.

This observed effect could potentially explain the overestimation of the beam size. One possible explanation for our overestimation of the beam size is that the value of $\frac{\Delta p_x}{\Delta K_2 L}$ which we have measured is too negative. If our measured

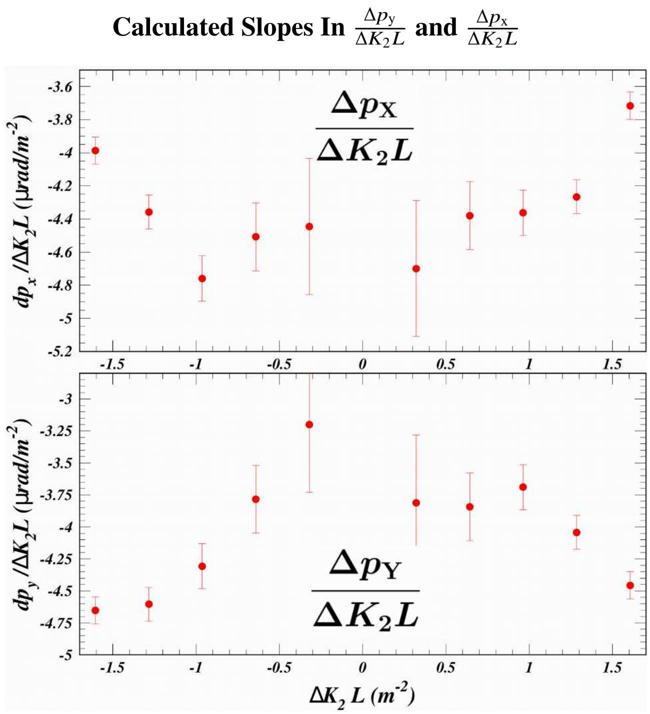

Figure 13: Measured values of the slopes $\frac{\Delta p_y}{\Delta K_2 L}$ and $\frac{\Delta p_x}{\Delta K_2 L}$ as functions of $\Delta K_2 L$ in scan 85. We would expect these measurements to be approximately constant in $\Delta K_2 L$, but they both show a clear dependence on $\Delta K_2 L$. The fact that these two values fluctuate similarly may suggest a relationship.

values of $\frac{\Delta p_x}{\Delta K_2 L}$ and $\frac{\Delta p_y}{\Delta K_2 L}$ are correlated as Figure 13 would suggest, then resolving the contradiction between our measured and expected $\frac{\Delta p_y}{\Delta K_2 L}$ will likely also have implications for our determination of $\frac{\Delta p_x}{\Delta K_2 L}$.

Figure 14 shows the calculated values of the slope, $\frac{\Delta p_x}{\Delta K_2 L}$, and the beam size, σ_x, over $\Delta K_2 L$ in scan 85. As expected, there is an obvious correlation between the beam size and the measurement of $\frac{\Delta p_x}{\Delta K_2 L}$, with less negative slopes leading to smaller beam sizes. In addition, it is clear that the beam size depends somewhat linearly on $|\Delta K_2 L|$, and decreases for larger $|\Delta K_2 L|$. This is consistent with the existence of an unaccounted for magnetic field, which impacts the measurement less for larger sextupole strength changes. Our expectation is that sufficiently large values of $\Delta K_2 L$ would yield relatively constant beam size measurements; however, we are limited in the value of $\Delta K_2 L$, with large values causing the beam to be lost in most cases.

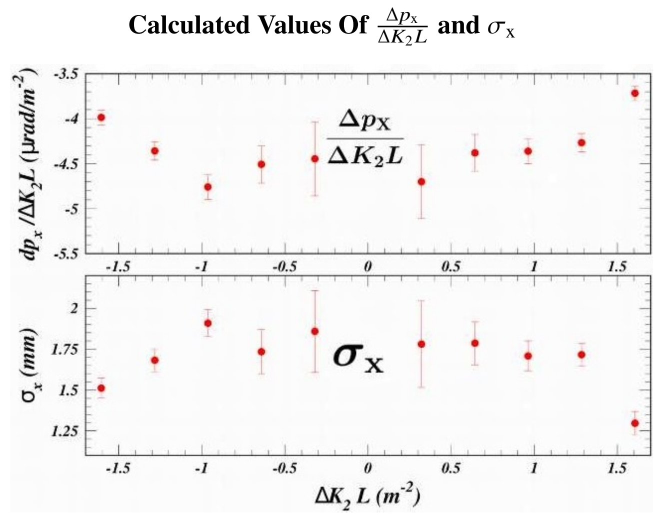

Figure 14: Calculated values of the slope, $\frac{\Delta p_x}{\Delta K_2 L}$, and the beam size, σ_x, over $\Delta K_2 L$ in scan 85. As expected, there is an obvious correlation between the beam size and the measurement of $\frac{\Delta p_x}{\Delta K_2 L}$, with less negative slopes leading to smaller beam sizes. In addition, it is clear that the beam size depends somewhat linearly on $\Delta K_2 L$, and decreases for larger $\Delta K_2 L$. This is consistent with the existence of an unaccounted for magnetic field, which impacts the measurement less for larger sextupole strength changes.

We plan to conduct further data scans to test for the effects of hysteresis, one candidate for the source of this disagreement in Δp_y. We believe that hysteresis is a reasonable candidate for this external effect because a change in $K_2 L$ of 1 m^{-2}, at a typical horizontal entrance position of $X_0 = 1$mm, corresponds to only about a 4 Gauss change in magnetic field strength, meaning that the measurement is particularly sensitive to small, unaccounted for magnetic fields.

CONCLUSION

In this paper, we derived expressions to calculate the beam size in sextupole magnets, while taking into account the finite length of the sextupole. These equations were then applied to data from the Cornell Electron Storage Ring, and the impact of correction terms related to the finite length correction was analyzed. We concluded that the finite length correction to our beam size measurement is negligible, and can be disregarded in general. In addition, the determination of error bars in our measurements was detailed, and the precision to which we can measure the beam size was discussed. It seems that we are capable in most cases of measuring the beam size to tenths of millimeters, or on the order of 10%. We do not see evidence that we are capable of leveraging statistical power to improve this measurement further. In addition, we identified a potential problem within our current analysis that suggests that there is an additional systematic effect we have not yet accounted for.

ACKNOWLEDGMENTS

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