Elementary Particle Physics I

Exercise Set 10: The Dirac Equation, QCD

1 Discussion Exercises

1.1 The Dirac Equation

We wish to derive a quantum field theoretical equation from the relativistic expression for the energy $E^2 = p^2 + m^2$. The Klein-Gordon equation has solutions with negative probability, because it is second-order in the time variable. In order to factorize the energy equation, let us suppose

$$p^\mu p_\mu - m^2 = (\beta^\kappa p_\kappa + m)(\gamma^\lambda p_\lambda - m)$$

where β^κ und γ^λ denote factors yet to be determined. Derive the anticommutation relations for β^κ and γ^λ. With these factors and the correspondence

$$\vec{p} \to \frac{1}{i}\nabla \qquad E \to i\frac{\partial}{\partial t}$$

one gets the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

where the γ are 4×4 matrices satisfying the anticommutation relations:

$$\{\gamma^\mu, \gamma^\nu\}{=}2g^{\mu\nu}$$

and are given by:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

where the σ^i are the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1.2 Angular Momentum Conservation and Spinors

This exercise will show that solutions to the Dirac equation carry intrinsic spin ($\vec{S}$) as well as orbital angular momentum ($\vec{L}$) and that the sum ($\vec{S} + \vec{L}$) is conserved.

1. Write the Hamilton Operator for the Dirac equation by solving it for p_0. The result is $H = \gamma^0(\gamma^i {\cdot} p^i + mc)$, where $1 \leq i \leq 3$ und $\vec{p} \equiv \frac{1}{i}\nabla$.

2. Calculate the commutator of H with the orbital angular momentum operator $\vec{L} \equiv \vec{r}\times\vec{p}$. (Result: $[H, \vec{L}] = -i\gamma^0\vec{\gamma} \times \vec{p}$.) Since it is not zero, we can conclude that orbital angular momentum alone is not conserved.

3. Now calculate the commutator of H with the spin operator:

$$\vec{S} \equiv \tfrac{1}{2}\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}.$$

Since the sum of the results from 2) and 3) is zero, we have shown that the total angular momentum is conserved.

1.3 Diagrams for Quark-Quark Scattering

Consider $p - \bar{p}-$scattering. We wish to calculate the relative contributions of the various subprocesses arising from constituent scattering processes. (We wish to take into account also constituents other than valence quarks.) The assumption of symptotic freedom means that we can consider the quarks and gluons to react as free particles.

1. Draw all possible lowest-order QCD diagrams.

2. Which of these diagrams have QED analogs?

2 Homework Exercises

2.1 Gamma Matrices

Show that the γ matrices given in the discussion exercise 1.1 satisfy the anticommutation relations derived in the exercise (3 points).

2.2 Time Dependence of Spinors

We will now consider the Dirac equation for the special case that the solutions $\psi(t)$ do not depend on position.

$$\tfrac{\partial\psi}{\partial x} = \tfrac{\partial\psi}{\partial y} = \tfrac{\partial\psi}{\partial z} = 0.$$

The correspondence

$$\vec{p} \to \tfrac{1}{i}\nabla \qquad E \to i\tfrac{\partial}{\partial t}$$

shows that this condition requires the momentum of the solutions to be zero. Show that the following are solutions of the Dirac equation (i.e. the Dirac spinors) can be written in the following way (2 points):

$$\psi^{(1)} = e^{-imt}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \psi^{(2)} = e^{-imt}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\psi^{(3)} = e^{+imt} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \psi^{(4)} = e^{+imt} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\psi^{(1)}$ und $\psi^{(2)}$ describe the two possible spin states of electrons. $\psi^{(3)}$ und $\psi^{(4)}$ describe the two spin states of positrons.

2.3 The Parity Transformation on Spinors

The parity transformation on spinors is $\psi \to \psi' = \gamma^0\psi$. Show that these solutions for fermions at rest are eigenstates of parity. What can you say about the relative parity of fermions and anti-fermions? (1 point)

2.4 The Renormalization Group Equation for QED and QCD

The renormalization of any field theory results necessarily in an energy-scale dependence of the coupling constant.

$$\alpha(Q^2) = \frac{\alpha(Q_0^2)}{1+b\cdot log(Q^2/Q_0^2)}$$

$QED : b = \frac{-\alpha(Q_0^2)}{3\pi}, Q_0 = m_e$ $\quad$ $QCD : b = \frac{\alpha(Q_0^2)}{12\pi}(33 - 2\cdot N_f), Q_0 = \Lambda \simeq 300\ MeV$

Where N_f is the number of flavours that can be produced for a given Q^2.

1. Calculate the relative change in the coupling constant for the following two cases (2 points):
   - QED: $Q^2 = m_e^2$ $\quad(\alpha(m_e^2) = 1/137)$ $\quad\to\quad$ $Q^2 = {m_Z}^2$
   - QCD: $Q^2 = m_{\Upsilon(4S)}^2$ $\quad(\alpha(m_{\Upsilon(4S)}^2) = 0.18)$ $\quad\to\quad$ $Q^2 = {m_Z}^2$

   Where $m_{\Upsilon(4S)}$ is the mass of the Υ(4S) meson ($b\bar{b}$ meson, $m_{\Upsilon(4S)}$=10.58 GeV)
   (Tip: you can consider the relative change of α^{-1})
2. In the previous calculation you have to use N_f=5. Why ? (1 point)
3. Discuss the meaning of the differing sign of the parameter b for QED and QCD. (1 point)