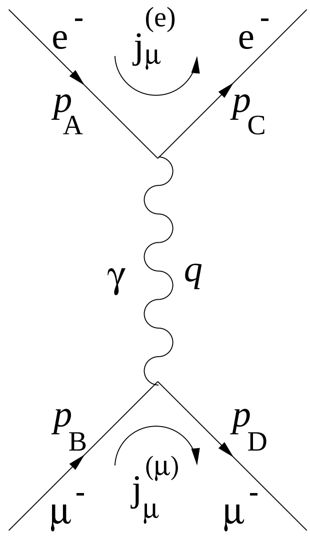
Elementary Particle Physics I

Exercise Set 5: Calculation of Scattering Cross Sections

1 Discussion Exercises

1.1 Feynman Rules for Spinless Electron-Muon Scattering

The goal of this exercise is to calculate the cross section for electron-muon scattering. In order to avoid some technical difficulties, we will assume the electron and muon to be spinless. In the diagram below, the time axis points horizontally to the right.



For a spinless "electron" ϕ_e we may use the solution to the Klein-Gordon equation.

$$\left(\partial_\mu \partial^\mu + m^2\right) \phi_e = 0. \tag{1}$$

As you know from electrodynamics, the coupling of the electron (charge $-e$, $e = \sqrt{4\pi\alpha}$) to the electromagnetic field $A_\mu = (A^0, \vec{A})$ is described by the substitution:

$$p^\mu \to p^\mu + eA^\mu. \tag{2}$$

- Carry out the substitution of Eq. 1 and determine the potential $V(x)$ with $\left(\partial_\mu \partial^\mu + m^2\right) \phi_e = -V\phi_e$. Ignore terms beyond leading order.

The transition amplitude is given by the matrix element, written in covariant form:

$$T_{fi} = -i \int \phi_C^*(x) V(x) \phi_A(x) d^4x. \tag{3}$$

- Write the transition amplitude for the potential in the form $T_{fi} = -i \int j_\mu^{(e)}(x) A^\mu(x) d^4x$. $j_\mu = (\rho, \vec{j})$ is the four-current of a scalar field. It satisfies the continuity equation.
- Show that $\phi(x) = Ne^{-ip\cdot x}$ is a solution to the Klein-Gordon equation and calculate the current $j_\mu^{(e)}$.

The electromagnetic field A^μ in which the electron scatters is produced by the muon in our case.

- Express the muon four-current $j^\mu_{(\mu)}$ as above for the electron and calculate the field A^μ produced by the muon according to the Maxwell equation $\partial_\nu \partial^\nu A^\mu = j^\mu_{(\mu)}$.
- Write the transition amplitude in the form

$$T_{fi} = -iN_A N_B N_C N_D (2\pi)^4 \delta^{(4)}(p_D + p_C - p_B - p_A)\mathcal{M}_{fi} \tag{4}$$

$$-i\mathcal{M}_{fi} = [ie(p_A + p_C)^\mu]\left[-i\frac{g_{\mu\nu}}{q^2}\right][ie(p_B + p_D)^\nu], \tag{5}$$

where $q \equiv p_D - p_B$. Describe the physical meaning of each of the terms and make a table of the Feynman rules derived in this manner. $\mathcal{M}$ is called the invariant matrix element.

2 Homework Exercises

2.1 Calculation of Spinless Electron-Muon Scattering (3 points)

Show that the matrix element for spinless electron-muon scattering can be expressed in the center-of-mass system while neglecting masses as

$$|\mathcal{M}_{\mathrm{e}^-\mu^-\to\mathrm{e}^-\mu^-}|^2 = e^4\left(\frac{3+\cos\theta}{1-\cos\theta}\right)^2 = e^4\left(\frac{2\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\right)^2. \tag{6}$$

The angle θ is the angle between the initial- and final-state electron momentum vectors.

2.2 Applying Crossing Symmetry (4 Punkte)

First show that

$$\mathcal{M}_{\mathrm{e}^-\mu^-\to\mathrm{e}^-\mu^-} = e^2\frac{u-s}{t}, \tag{7}$$

where $s = (p_A + p_B)^2, t = (p_A - p_C)^2, u = (p_A - p_D)^2$.

The calculation of the matrix element for this process can be done, of course, using the Feynman rules derived in the discussion exercise. It is a lot easier, however, to do it using crossing symmetry, as described below.

Draw the Feynman graphs for

$$\mathrm{e}^-(p_A)\mu^-(p_B) \to \mathrm{e}^-(p_C)\mu^-(p_D)$$

and

$$\mathrm{e}^+(k_B)\mathrm{e}^-(k_A) \to \mu^+(k_C)\mu^-(k_D)$$

The quantities in parentheses designate the four-momenta to be included in the graphs.

As you see, these graphs are the same when one replaces the anti-particles with particles of opposite four-momentum. Write down the relationships between the four four-momenta in the two graphs. Define the Mandelstam variables and compare them for the two graphs. What is the relationship between $\mathcal{M}_{\mathrm{e}^+\mathrm{e}^-\to\mu^+\mu^-}$ and$\mathcal{M}_{\mathrm{e}^-\mu^-\to\mathrm{e}^-\mu^-}$?

2.3 Phase Space for Two-Body Scattering (3 points)

Transition rates are calculated according to Fermi's Golden Rule as follows:

$$\Gamma = 2\pi |M|^2 \times (PhaseSpace)$$

The mysteries of the dynamics are hidden in $|M|^2$ and the phase space factor (density of final states) is determined by kinematics. We consider two-body scattering and calculate the cross section. The transition rate is given by the product of the cross section and the flux factor Φ:

$$\Gamma = \Phi \times \sigma$$

where

$$\Phi = \frac{n_A n_B}{V}(|v_A| + |v_B|), \; v = \frac{|\vec{p}|}{E}.$$

V is a hypothetical volume in which the interaction takes place. It must cancel in the final result, so we set it to unity and ignore it. We ignore n_A and n_B as well, since we desire the scattering cross section for individual particles. The differential transition rate for producing final-state particles with momenta p_C und p_D is given by

$$d\Gamma = |M|^2 \cdot \frac{1}{4E_A E_B} \cdot dP_{CD} \cdot S,$$

where the factor $\frac{1}{4E_A E_B}$ serves to normalize the initial-state wave functions (It is sometimes considered to be part of the definition of the flux factor.), and the Lorentz-invariant phase space factor is

$$dP_{CD} = (2\pi)^4 \delta^4(p_A + p_B - p_C - p_D) \frac{d^3 p_C}{(2\pi)^3 2E_C} \frac{d^3 p_D}{(2\pi)^3 2E_D}.$$

The delta function expresses the conservation of energy and momentum. The factor S represents the sum over final-state spin states:

$$\mathrm{S} = (2s_C + 1)(2s_D + 1)$$

In the case the particles C and D are identical, S also includes a factor $\frac{1}{2!}$.
Calculate the integral over $d^3 p_C$ und $d^3 p_D$ in order to get the differential cross section $\frac{d\sigma}{d\Omega}$. The result is

$$\frac{d\sigma}{d\Omega} = \frac{1}{16(2\pi)^2} \frac{|M|^2}{(E_A + E_B)^2} \frac{|p_C|}{|p_A|} S.$$

Guidance :
Express $d\sigma$ as a function of $d\Gamma$!
Use the relationship $\delta^4(p) = \delta(E)\delta(\vec{p})$ and calculate the integral over $d^3 p_D$! (CMS!!!)
Replace $d^3 p_C$ by spherical coordinates, define $E_C + E_D = E$ and replace the integration over dp_C by an integration over dE by calculating $\frac{dE}{dp_C}$!
Calculate the integral over dE!