Particle Physics I

Exercise Set 7: Isospin and Hypercharge

1 Discussion Exercises

1.1 Isospin

The proton p and the neutron n have almost the same mass and behave very similarly under strong interaction. The same is true for the pions $\pi^+\pi^0\pi^-$. If we consider for instance the following reaction of pion-scattering off the deuteron d

$$\begin{aligned} \pi^+ + d &\rightarrow p + p && (1)\\ \pi^0 + d &\rightarrow p + n && (2)\\ \pi^- + d &\rightarrow n + n, && (3) \end{aligned}$$

we find that $\sigma_{\pi^+d\rightarrow pp} = \sigma_{\pi^-d\rightarrow nn}$. It seems therefore obvious to consider the proton and the neutron as two states of one particle, the nucleon N. In the same way, the three pions can be interpreted as three different states of one particle. The strong interaction is, however, not completely independent of the charge states since $\sigma_{\pi^0d\rightarrow pn} = \frac{1}{2}\sigma_{\pi^+d\rightarrow pp}$. Furthermore, the deuteron is a bound pn state but there is no bound pp or nn state. A simple additive quantum number to differentiate between the several states is hence not sufficient.
The different nucleon (or pion) states are described within the isospin formalism as eigenstates of a further (postulated) spin-like degree of freedom: the isospin.

- Assign the isospin quantum numbers to the nucleon doublet, the pion triplet and to the deuteron.
- What are the operators which generate transitions within the members of a multiplet? Show this explicitly for the doublet case.

Instead of charge indepence, we only require isospin invariance (i.e. the effect of the strong interaction is invariant under rotation in isospin space). This is equivalent to the claim that the operator of the strong interaction commutes with these rotations. These rotations form the $SU(2)$-group, which is the group of special unitary transformations in two dimensions. The multiplets are representations of this group ($SU(2)$ multiplets). You should be familiar with the doublets (fundamental representation), where the generators of $SU(2)$ are the Pauli matrices. Two-particle reactions and bound states are described by combined representations (or product representations).

- What are the eigenstates of the operators $\vec{I}^2$ and I_3 of a system of two spin-$\frac{1}{2}$-particles, if $\vec{I} = \vec{I}_1 + \vec{I}_2$?

This combined representation can be expressed as irreducible representations. These irreducible representations can be created with the group generators of any of its members. In this case the combination and the decomposition is written as

$$2 \otimes 2 = 3_S \oplus 1_A. \tag{4}$$

- Discuss this notation.

The isospin is handled in the same way.

- What are the four isospin states of a nucleon-nucleon system?
- What are the isospin states of the pion-deuteron system of eq. (1)-(3)?
- Calculate the ratios of cross sections of reactions (1)-(3).
- How can the isospin invariance explain the same masses within a multiplet?
- Is isospin invariant under all interactions?

1.2 Hypercharge

The $SU(2)$ description of isospin became insufficient with the discovery of "strange" particles. The long lifetime of these particles could only be explained by introducing a new conservation law forbidding the decay via the strong interaction, so that these particle could only decay via the weak interaction. This conserved quantity was called *strangeness*:

$$S = 0 \quad : \quad \pi, N, \Delta, \eta, \eta' \tag{5}$$
$$S = +1 \quad : \quad K^+, K^0 \tag{6}$$
$$S = -1 \quad : \quad \Lambda, \Sigma, K^-, \overline{K^0}. \tag{7}$$

Instead of the strangeness, *hypercharge* $Y \equiv B + S$ is frequently used, where B is the baryon number.

- Draw the pseudoscalar mesons in a I_3-Y-diagram.

These mesons are composed of u-, d- und s-quarks.

- What is the isospin and hypercharge of these quarks? Draw u,d,s and $\bar{u}$,$\bar{d}$,$\bar{s}$ in a I_3-Y-diagram.

The $SU(2)$ of isospin and the $U(1)$ of hypercharge can be combined in $SU(3)_{flavour}$. Its fundamental representations "3" und "$\bar{3}$" are uds und $\bar{u}\bar{d}\bar{s}$-triplets. You just have drawn them.

- Build graphically the pseudoscalar meson nonet out of these triplets.

It can be shown that $3 \otimes 3 = 8 \oplus 1_S$. The fact that all observed particles can be classified into such multiplets and that each position in the multiplet was assigned to a particle, was the the strongest argument in favour of the quark model.

- Show now that the Gell-Mann-Nishijima relation

$$Q = I_3 + \frac{1}{2}Y \tag{8}$$

is valid for quarks.

Since all quantum number in this equation are additive, the Gell-Mann-Nishijima relation is a trivial consequence of the quark model.

2 Homework

2.1 Isospin in Pion-Nucleon Scattering (5 Points)

The $\Delta(1232)$-resonance appears in four charged states $\Delta^{++}, \Delta^{+}, \Delta^{0}, \Delta^{-}$. They are hence isospin 3/2 states. Each of these states decays mainly into a pion and a nucleon.

- Write down these decay channels.
- Calculate the branching ratios by using the isospin formalism.

We now consider pion-nucleon scattering with production of a resonant $I = 3/2$ intermediate state. This is indeed in the region of the $\Delta(1232)$ resonance the dominant contribution to the cross section.

- Show under the assumption of isospin invariance of the strong interaction that

$$\sigma_{\pi^+p\to\pi^+p} : \sigma_{\pi^-p\to\pi^-p} : \sigma_{\pi^-p\to\pi^0n} = 1 : 1/9 : 2/9. \tag{9}$$

The Clebsch-Gordan coefficients can be found in the table.

2.2 The anomalous magnetic moments of the nucleons (5 Points)

The flavour-spin wave function of the proton is given by

$$|p\uparrow\rangle = \frac{1}{\sqrt{18}}\,|u\uparrow u\downarrow d\uparrow + u\downarrow u\uparrow d\uparrow - 2u\uparrow u\uparrow d\downarrow + Perm(udu) + Perm(duu)\rangle\,. \tag{10}$$

- Write down the flavour-spin wave function $|n\uparrow\rangle$ of the neutron.

Assume that the magnetic moments of the nucleons can be calculated as the expectation values of the operator $\hat{\mu}_N = \sum \hat{\mu}_q$, where $\hat{\mu}_q$ are the operators of the quarks' magnetic moments. All quarks in the nucleon are summed up. Note that $\hat{\mu}_N$ is spin sensitive.

- Show that

$$\mu_p = \tfrac{1}{3}(4\mu_u - \mu_d) \tag{11}$$

$$\mu_n = \tfrac{1}{3}(4\mu_d - \mu_u). \tag{12}$$

Remember that the magnetic moment of a point-like fermion f is given by $\mu_f = Q_f/2m$, where Q_f is the charge of the fermion.

- Calculate the ratio μ_n/μ_p, by assuming $m_u = m_d$. Compare this value to the measured one ($\mu_n/\mu_p = -(0.68497945 \pm 0.00000058)$.