E821 Differential Decay Correction

Brandon Krouppa[1,2], Eric Metodiev[1,3], and William Morse[1]

[1]BNL, [2]SULI Program, [3]SHLI Program

November 28, 2012

The E821 differential decay correction has four contributions: Francis' g-2 note 34 correction of < +0.03*ppm*; *Brandon's kicker correction of* < +0.06*ppm, and Eric's beamline estimate of* ±0.045*ppm. For the latter, Volodia will verify Eric's (blinded) sign with his E821 beamline simulation ("trust, but verify"). Also, Eric pointed out that the higher energy muons give higher energy electrons, and these take longer to reach the detectors. A first guestimate has been done. The above estimates are all small compared to the E821 systematic error, but large compared to the FNAL E969 systematic error budget.*

I. Introduction

Francis Farley wrote g-2 note 34 *g-2 Phase and Mu-Momentum – Differential Decay Correction* in 1989. "Whether the g-2 phase varies with time because of differential decay: the (high momentum) muons on the outside live longer." Francis guestimated a correction of < +0.03*ppm*, i.e., we measure a lower frequency, and so must correct it upwards. We need a real simulation to verify Francis' guestimate. The sign has been checked by several people, but more would be better (in E821 we got the kicker magnet polarity wrong, even though it was checked). Anyway, Francis' differential decay correction was then largely forgotten. I re-discovered it last year while thinking about the new FNAL experiment, and then realized that there would also be differential decay corrections due to a non-perfect kicker, pion decays in the bend sections of the beamline, etc. Brandon studied the former effect during the summer, and Eric the latter.

II. E821 Kicker Differential Decay – Brandon's Estimate

Fig. 1 shows the E821 kicker pulse [1]. Brandon's program followed the muons from the end of the inflector to the kicker, including Coulomb multiple scattering in the outer Q1 electrode and HV standoffs, if appropriate. He then followed the muon through nine turns getting kicks as shown in Fig. 1. The (*x*,*y*) dependence of the kicker field was taken from refs 2-3. Scraping was not included in the program, nor were collimator survey errors, non-uniform beta functions, etc. Brandon used the average value of the beta functions (see Fig. 2).

Fig. 3 shows the horizontal phase space *x*-*x*' plot from Hugh Brown's file of muons at the end of the inflector (black plus marks), and those muons which survive nine turns without hitting collimators (red crosses). A general feature of these plots is that there are more stored muons with negative *x*' than positive *x*'. Someone needs to study the effect

on the number of stored muons from changing the inflector angle for the FNAL experiment.

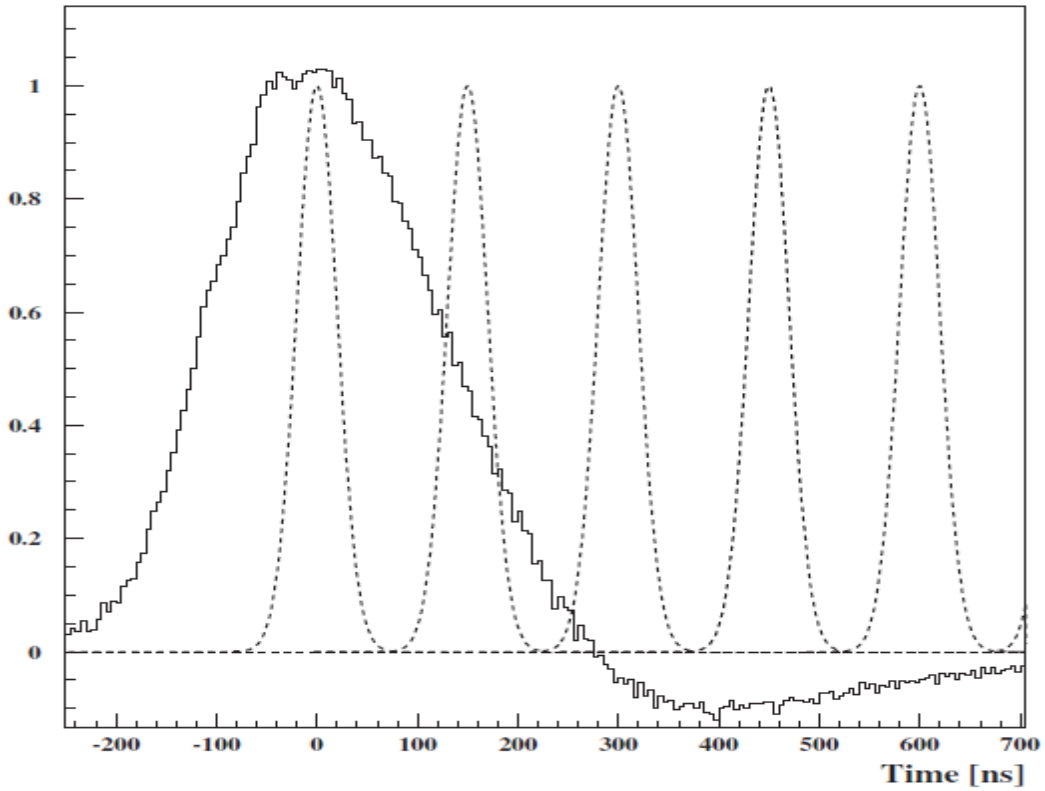


Fig. 1. Kicker pulse with a schematic representation of the bunch circulating with period 149ns.

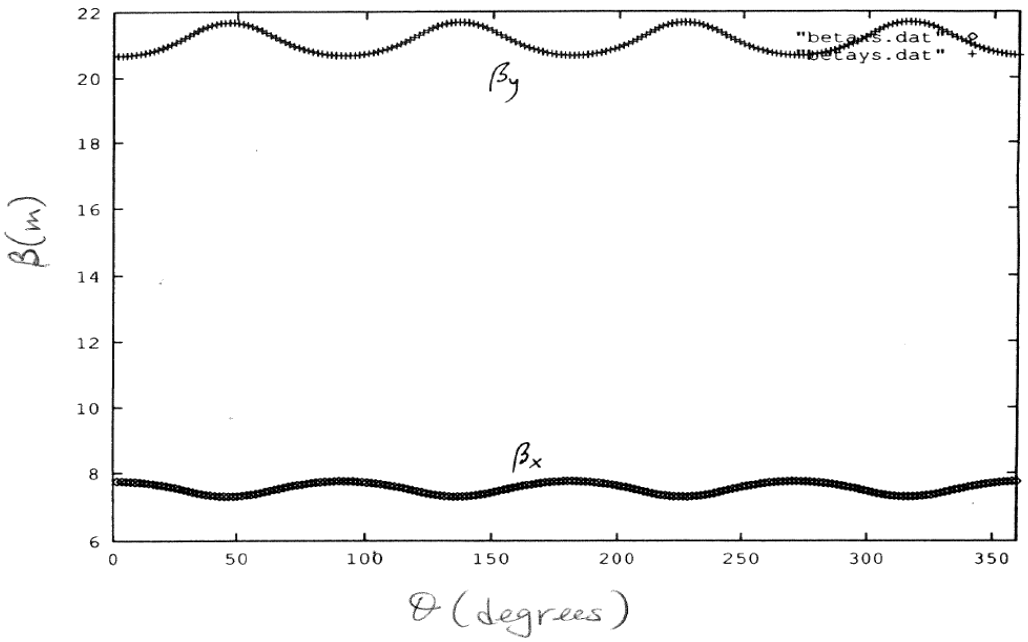


Fig. 2. E821 beta functions from the Design Report.

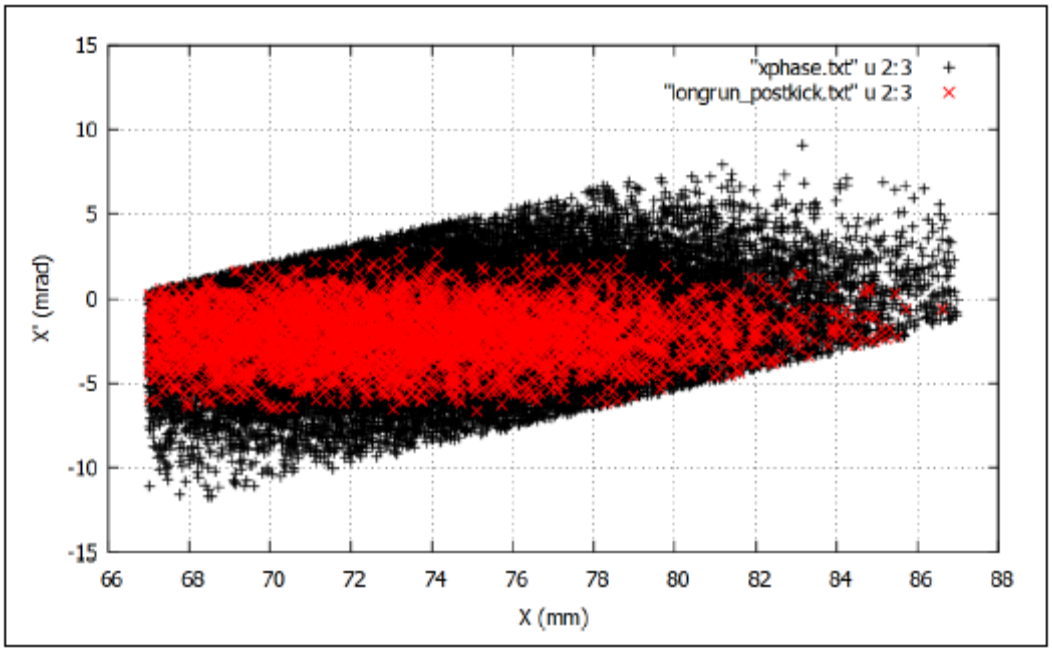


Fig. 3. Horizontal phase space plot at the end of the inflector (black plus marks), and those muons which survive nine turns (red crosses).

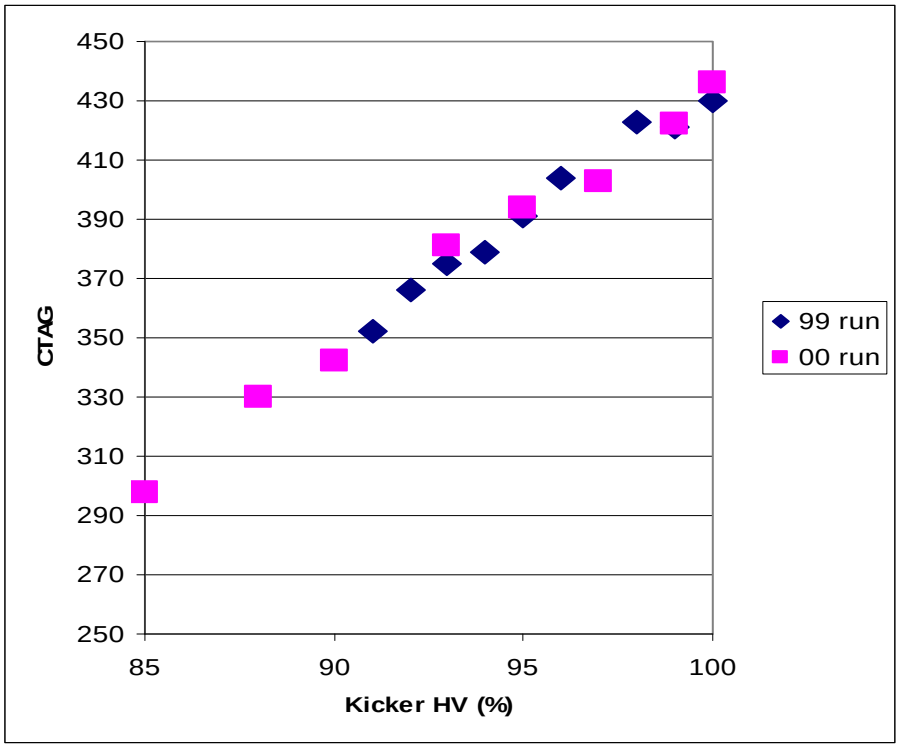


Fig. 4. Observed number of stored muons (a.u.) vs. kicker HV. 100% on the three kicker HV control modules was set to be 86KV, 91KV, and 93KV, respectively.

Fig. 4 shows the observed number of stored muons vs. kicker HV. Fig. 5 shows the simulation of the number of stored muons vs. the first turn kick at the center of the kicker (red plus marks). Also shown in blue stars is the data from Fig. 4, assuming 90KV average kicker HV gives first turn kick strength in the middle of the kicker of 6.2mrad. We can say from this plot that the first turn kick strength in the center of the kicker is less than 6.2mrad. It turns out (see below) that the kicker differential decay systematic effect depends on the kick strength, and is larger for higher kick strengths. Thus in order to set an upper limit on the systematic effect, we want to set an upper limit on the kick strength.

Ref. 3 gives from the laser measurement of the straight R&D kicker field = 165G at the peak with ±15% absolute error. This was at (x, y) = (25mm, 0) for 95KV. From ref. 4, for the 1999 run, 100% on the three kicker HV control modules was set to be 86KV, 91KV, and 93KV, respectively, for an average of 90KV. For higher voltages we had excessive sparking, which we didn't have on the R&D kicker. We then guestimate for E821 a first turn kick at (x, y) = (0, 0) for 100% on the 1999 kicker HV control modules of:

$$\frac{3\times 1.6m}{7.11m}\frac{0.0165T}{1.45T}\frac{90KV}{95KV}\frac{191}{207}\cos\left(\frac{\pi}{4}\right)=(6.2\pm 1.0)mrad \qquad (1)$$

The cos term is a guestimate for the efficiency for a distributed kick, instead of a delta function kick at 90 degrees phase advance. We seem to be seeing from Fig. 4 somewhat less than expected from the R&D kicker measurements. This was never properly measured for the real kickers, only for the R&D kicker. Obviously, this must be done for the real FNAL kicker.

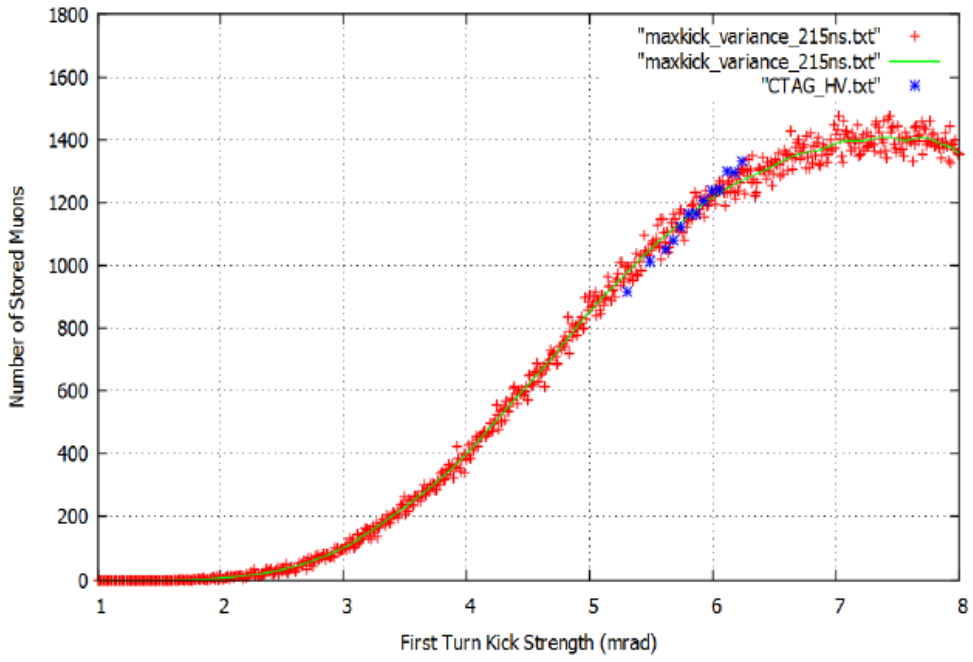


Fig. 5. Simulated number of stored muons (a.u.) vs. first turn kick at x = y = 0 (red plus marks), and observed in E821 (blue stars).

Fig. 6 shows the simulated equilibrium radius distribution for nine turns for 6.2 mrad first turn kick at the center of the kicker, and the E821 observed distribution for >100 turns. Higher kick strength moves the distribution to lower $x_{e,}$ so we can say that the kick strength was not greater than 6.2 mrad. Fig. 7 shows the simulated and observed kicker timing curve for 6.2 mrad kick. The simulated timing curve is wider than the observed timing curve. A smaller kick gives a narrower timing curve, so we can again say that the kick strength was not greater than 6.2 mrad.

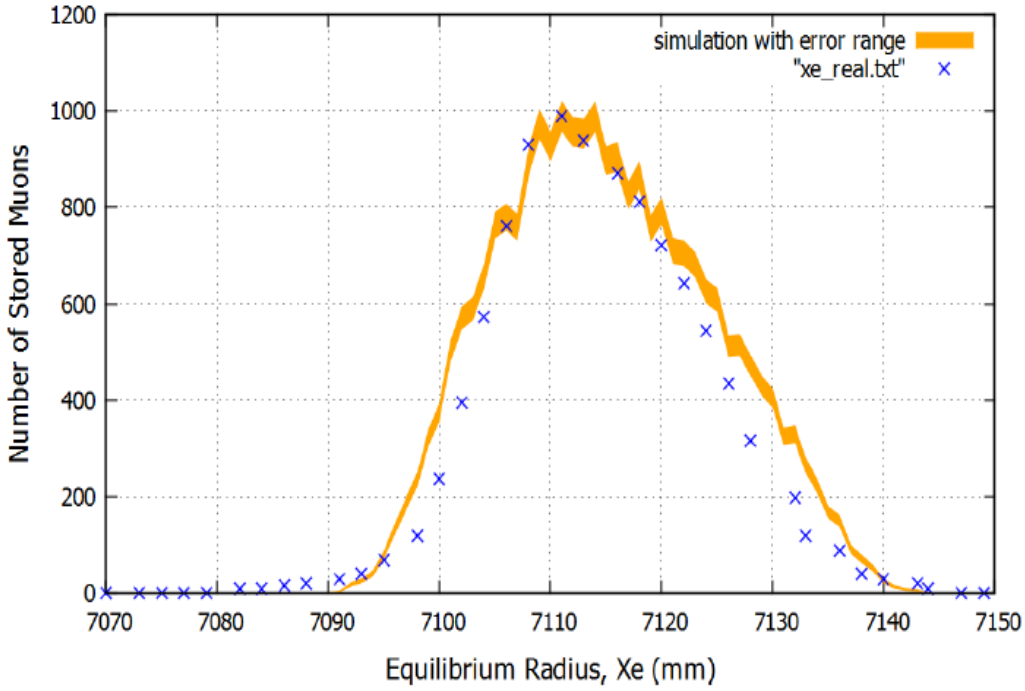


Fig. 6. Simulated for nine turns, and observed after 100 turns.

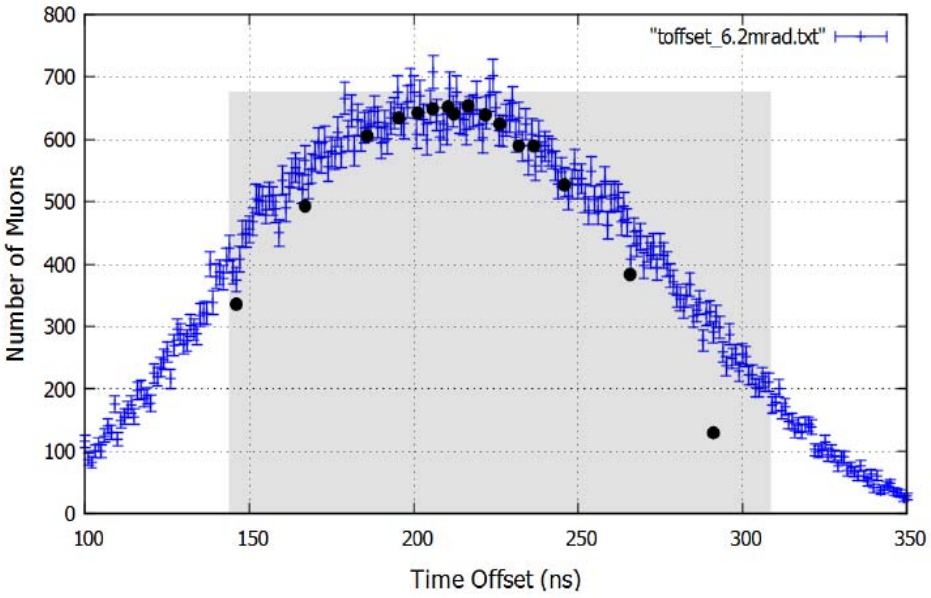


Fig. 7. 6.2mrad simulated (blue points) and E821 observed (black dots) kicker timing curve.

Next we calculate the kicker differential decay effect. The functional form for muon decays is [1]:

$$f(t) = \frac{N}{\gamma\tau}\exp\left(\frac{-t}{\gamma\tau}\right)\left[1 + A\cos(\omega t + \phi)\right] \qquad (2)$$

where $\tau = 2.197\mu s$ and ω is the anomalous magnetic moment frequency we wish to measure to high precision. Consider a muon bunch which comes into the kicker centered at $\phi = 0$, and ignore the $1 + A$ and N terms in order to simpify the formulae. Writing the bunch head and tail populations with different g-2 phases ωt_k, where t_k is the time the muon goes through the kicker with respect to the center of the bunch, showing the signs explicitly:

$$f(t) = \frac{1}{2\gamma\tau}\left[\exp\left(\frac{-t}{\gamma\tau}\right)\cos(\omega t + \omega\,|\,t_k\,|) + \exp\left(\frac{-t}{\gamma\tau}\right)\cos(\omega t - \omega\,|\,t_k\,|)\right] \qquad (3)$$

$$\frac{df}{f} = \left(\frac{t}{\gamma\tau} - 1\right)\frac{d\gamma}{\gamma} \qquad (4)$$

$$f(t) \approx \frac{1}{2\gamma_0\tau}\exp\left(\frac{-t}{\gamma_0\tau}\right)\left[\left(\left[1 - \frac{t}{\gamma_0\tau}\frac{d\gamma}{\gamma_0}\right]\cos(\omega t + \omega\,|\,t_k\,|) + \left[1 + \frac{t}{\gamma_0\tau}\frac{d\gamma}{\gamma_0}\right]\cos(\omega t - \omega\,|\,t_k\,|)\right)\right] \qquad (5)$$

$$= \frac{1}{\gamma\tau_0}\exp\left(-\frac{t}{\gamma_0\tau}\right)\left[\cos(\omega t)\cos(\omega t_k) + \varepsilon\right] \qquad (6)$$

$$\varepsilon = \frac{t}{\gamma_0\tau}\frac{d\gamma}{\gamma_0}\sin(\omega t_k)\sin(\omega t) \qquad (7)$$

$$\cos(\omega t_k) \approx 1 \qquad \sin(\omega t_k) \approx \omega t_k \qquad (8)$$

$$f(t) \approx \frac{1}{\gamma_0\tau}\exp\left(\frac{-t}{\gamma_0\tau}\right)\left[\cos(\omega t) + \frac{t}{\gamma_0\tau}\frac{d\gamma}{\gamma_0}\omega t_k \sin(\omega t)\right] \qquad (9)$$

$$\approx \frac{1}{\gamma_0\tau}\exp\left(\frac{-t}{\gamma_0\tau}\right)\cos\left(\omega t\left(1 - \frac{t_k}{\gamma_0\tau}\frac{d\gamma}{\gamma_0}\right)\right) \qquad (10)$$

Thus the kicker differential decay systematic error is:

$$\frac{d\omega}{\omega} \approx < \frac{t_k}{\gamma_0 \tau} \frac{d\gamma}{\gamma_0} > \qquad (11)$$

Equ. 11 differs from the formula used in my earlier kicker differential decay g-2 note. Hopefully it is now correct. Equ. 11 can be evaluated from Brandon's simulation result shown in Fig. 8, integrated over the bunch width. We get < +0.06ppm kicker differential decay systematic effect. The observed E821 frequency must be increased due to this systematic effect.

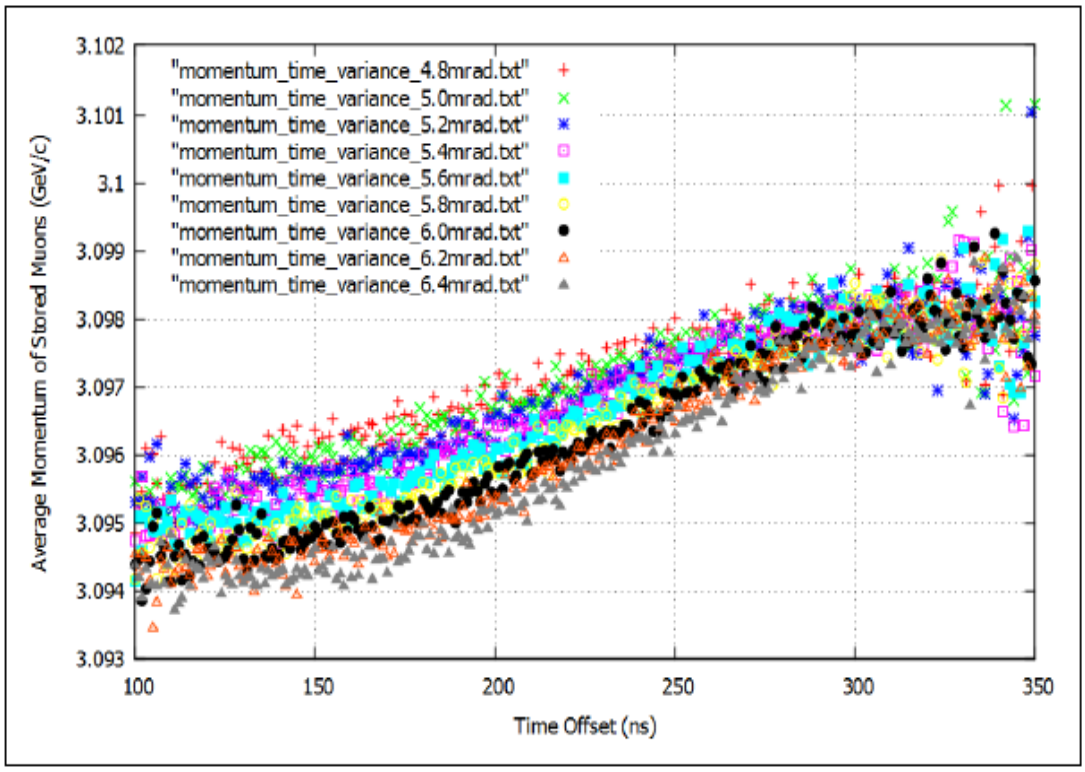


Fig. 8. Correlation between the momentum and kicker time offset from the beginning of the kicker pulse (see Fig. 7) for different first turn kicks. The muon bunch center is around 215ns in this plot.

III. Beamline Differential Decay - Eric

Next we turn to the momentum-spin correlation coming from the pion/muon beamline. Eric plotted this from a file of 90K simulated muons at the end of the inflector from Hugh Brown via Paul Debevec (see Figs. 9-11).

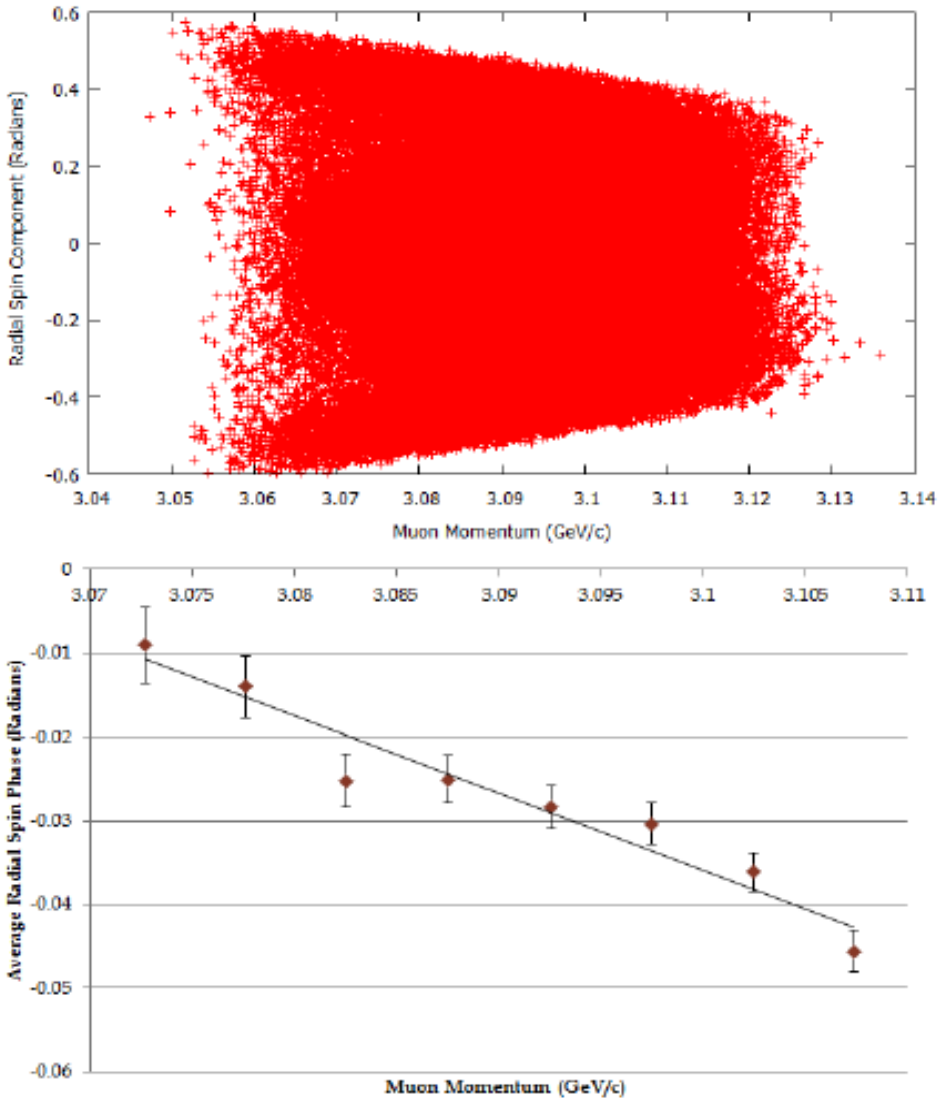


Fig. 9. Radial component of the muon spin vs. momentum from Hugh Brown's file of muons at the end of the inflector.

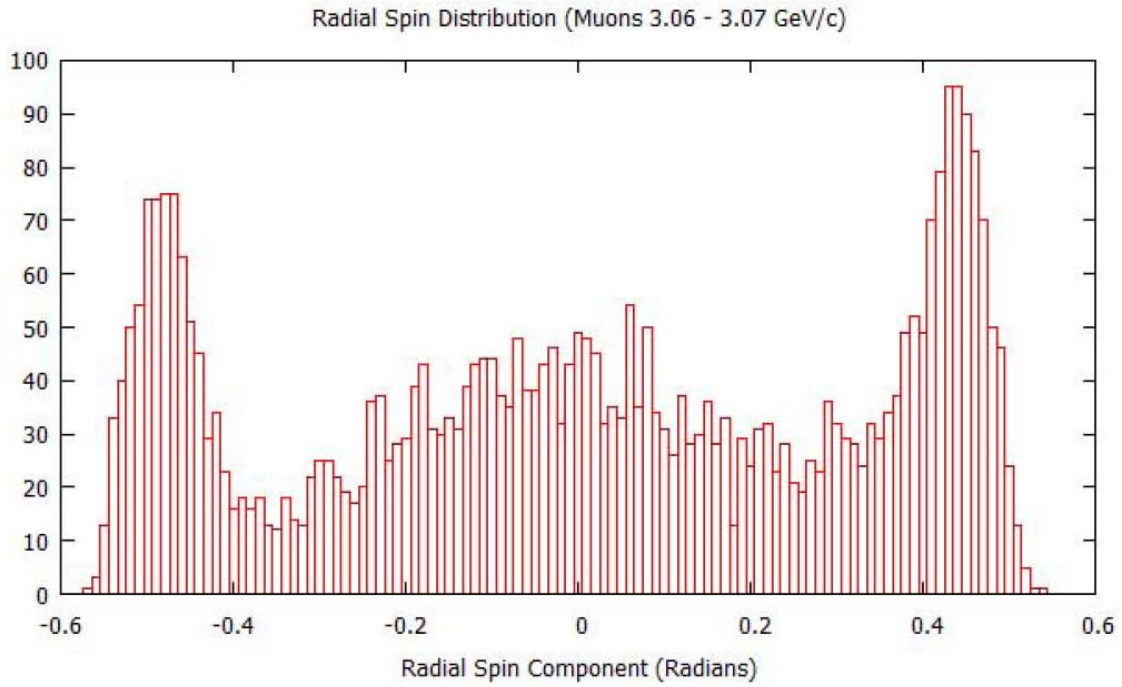


Fig. 10. Radial component of the muon spin for muon momentum 3.06 – 3.07 GeV/c.

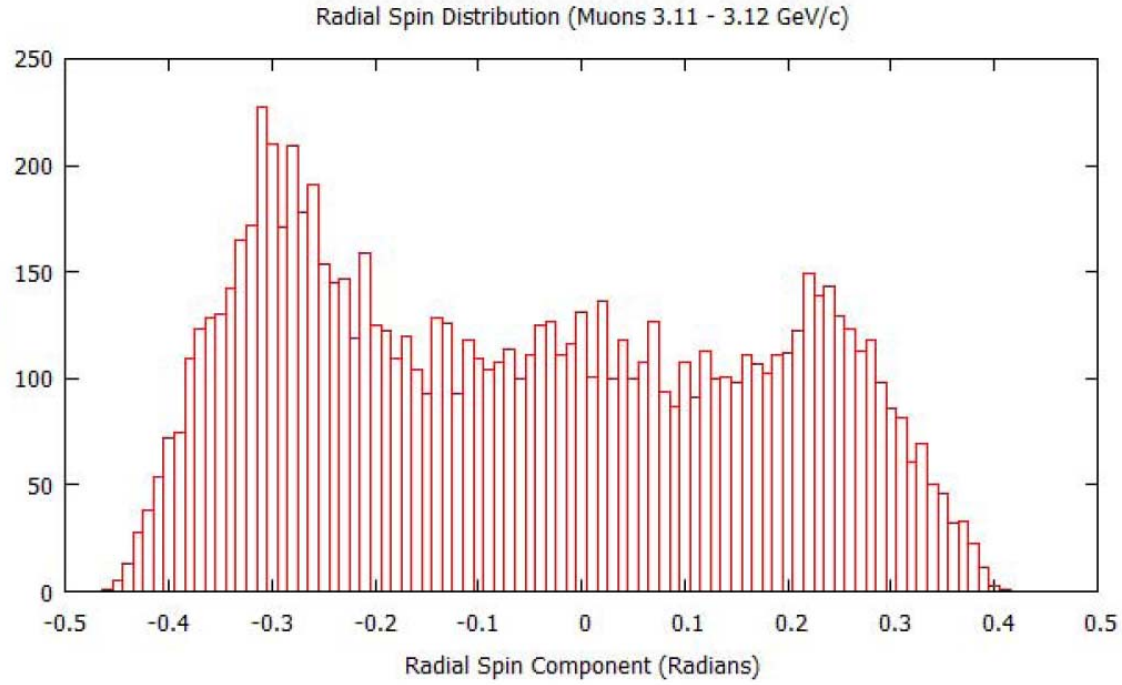


Fig. 11. Radial component of the muon spin for muon momentum 3.11 – 3.12 GeV/c.

2 Systematic Error

We start with the five parameter function:

$$f(t) = \frac{dN}{dt} = \frac{N}{\gamma\tau} \cdot \exp\left(\frac{-t}{\gamma\tau}\right)[1 + Acos(\omega_a t + \phi)]\,. \tag{2}$$

The problem arises when the average radial spin phase is correlated with the muon momentum ($\bar{\phi}_x = \bar{\phi}_x(p)$). This would not be a problem if the average momentum did not change with time; however, due to differential decay, momentum changes over time. To first order, we can express this correlation with $\phi = \phi_0 + t\frac{d\phi}{dt}$. Substituting this expression in Equation 2 yields:

$$f(t) = \frac{dN}{dt} = \frac{N}{\gamma\tau} \cdot \exp\left(\frac{-t}{\gamma\tau}\right)\left[1 + Acos(\omega_a t + \phi_0 + t\frac{d\phi}{dt})\right] \tag{3}$$

$$f(t) = \frac{dN}{dt} = \frac{N}{\gamma\tau} \cdot \exp\left(\frac{-t}{\gamma\tau}\right)\left[1 + Acos(\left(\omega_a + \frac{d\phi}{dt}\right)t + \phi_0)\right]. \tag{4}$$

The measured precession frequency ω_M is the coefficient of the time-dependent term in the cosine in Equation 4:

$$\omega_M = \omega_a + \frac{d\phi}{dt} \tag{5}$$

Therefore, the actual precession frequency is related to the measured precession frequency by:

$$\omega_a = \omega_M - \frac{d\phi}{dt} \tag{6}$$

The correction due to the spin-momentum correlation differential decay is then:

$$\Delta\omega = \frac{d\phi}{dt}. \tag{7}$$

The precise correlation between spin and momentum $\frac{d\phi}{dp}$ is currently being investigated by simulation. The changing average momentum over time to first order $\frac{dp}{dt}$ is necessary to be able to relate this to the systematic error by:

$$\frac{d\phi}{dt} = \frac{d\phi}{dp} \cdot \frac{dp}{dt}. \tag{8}$$

3 Differential Decay Simulation

In order to find $\frac{dp}{dt}$ to satisfy Equation 8, a gaussian distribution with $\sigma_\gamma = 0.03564$[2] was iterated to decay over several muon lifetimes. The changing average γ value was studied, because differential decay rates can be related to γ by:

$$\frac{df}{f} = \left(\frac{t}{\gamma\tau} - 1\right)\frac{d\gamma}{\gamma}, \tag{9}$$

where $f = \frac{dN}{dt}$.

The resulting γ dependence on time took on the linear form shown in Figure 2.

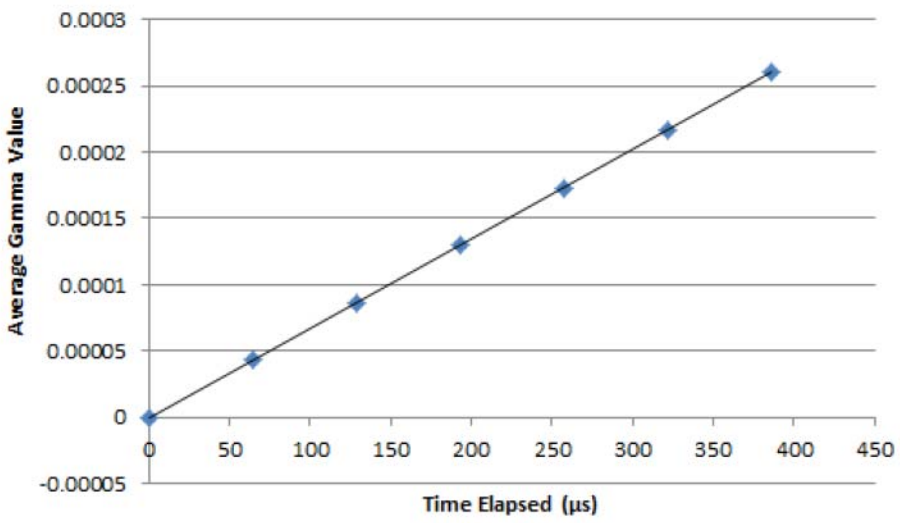


Figure 2: The average γ value over time for a simulated gaussian distribution of muons with $\sigma_\gamma = 0.03564$ over 6 lifetimes.

The instantaneous rate of change of gamma with respect to time shown in Figure 2 was calculated to be:

$$\frac{d\gamma}{dt} = 6.73 \times 10^{-7} \mu s^{-1}. \tag{10}$$

Relating this to the momentum time dependence by $\frac{dp}{dt} = m_\mu \beta \frac{d\gamma}{dt}$ yields the value:

$$\frac{dp}{dt} = 7.06 \times 10^{-8} \ \frac{GeV}{c} \mu s^{-1}. \tag{11}$$

Using this value for $\frac{dp}{dt}$ and applying Equation 8 yields the radial spin phase dependence on time:

$$\Delta\omega = \frac{d\phi}{dt} = 6.522 \times 10^{-8} \ rad \ \mu s^{-1}. \tag{12}$$

The systematic error resultig from this value is:

$$\frac{d\omega}{\omega} = 0.0453 \ ppm. \tag{13}$$

4 Analytic $d\gamma/dt$ Estimation

We begin by assuming that there is a Gaussian distribution of gamma values with $\sigma = 0.356$. The distribution of muon gammas would therefore take the form:

$$N(\gamma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\frac{1}{2}(\gamma-\gamma_0)^2}{\sigma^2}} \quad (14)$$

A muon with relativistic γ decays according to $N(t) = exp(-t/\gamma\tau)$ with the mean rest lifetime of the muon $\tau = 2.197\mu s$. Taking these different decay rates into account yields the overall dependence:

$$N(\gamma, t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\frac{1}{2}(\gamma-\gamma_0)^2}{\sigma^2} - \frac{t}{\gamma\tau}} \quad (15)$$

To find the average property of any distribution $D(x)$, we can use the expression:

$$\bar{x} = \frac{\int_{-\infty}^{\infty} D(x) x\, dx}{\int_{-\infty}^{\infty} D(x) dx} \quad (16)$$

Applying this to find $\bar{\gamma}$ for some time t yields:

$$\bar{\gamma} = \frac{\int_{-\infty}^{\infty} N(\gamma, t)\gamma d\gamma}{\int_{-\infty}^{\infty} N(\gamma, t) d\gamma} \quad (17)$$

More explicitly:

$$\bar{\gamma} = \frac{\int_{-\infty}^{\infty} \gamma e^{-\frac{\frac{1}{2}(\gamma-\gamma_0)^2}{\sigma^2} - \frac{t}{\gamma\tau}} d\gamma}{\int_{-\infty}^{\infty} e^{-\frac{\frac{1}{2}(\gamma-\gamma_0)^2}{\sigma^2} - \frac{t}{\gamma\tau}} d\gamma} \quad (18)$$

Tabulated values:

# of Lifetimes	0	5	10	20
Total Time (μs)	0	322	644	1288
Average γ	29.3	29.300216	29.300434	29.300867

These points form a line that precisely agrees with the slope yielded by simulation of:

$$\frac{d\gamma}{dt} = 6.73 \times 10^{-7}\, \mu s^{-1} \quad (19)$$

IV. Electron Energy Differential Decay

Eric also pointed out another differential decay effect: the higher energy muons give higher energy electrons, and these take longer to reach the detectors (see Fig. 1). I did a phase space calculation of this systematic effect for decay electrons close to the endpoint.

For E821 we stored muon *dp/p* HWHM ≈ ±0.25%. Figs.2a-b show the *x*-*x*' phase space plots for a muon on it's equilibrium orbit with *dp/p* = 0.25% (Fig. 2a) and *dp/p* = -0.25% (Fig. 2b) decaying to electrons at the end point, lower by 0.06%, 0.12%, etc. until it has betatron oscillation with $x = \rho - 711.2cm < -5cm$, at which point it quickly hits either the quad electrode, kicker electrode, etc. The decay electron end point is:

$$p_{e,\max} \approx p_\mu \left(1 + \frac{m_\mu^2}{4p_\mu^2}\right) \approx 1.0003 p_\mu \qquad (1)$$

From Fig. 2a, such decay electrons with *dp/p* > -0.4% are "stored". With *dp/p* < -0.4%, they go through material, lose energy, and strike a detector. From Fig. 2b, such decay electrons with *dp/p* > -0.2% with are "stored".

Electrons lose energy due to synchrotron radiation. The fractional energy loss per turn is:

$$\frac{\Delta E}{E} = 8.858 \times 10^{-5} \frac{m}{GeV^3} \frac{E^3}{\rho} \approx 0.037\% \qquad (2)$$

Thus the higher energy muons produce more electrons which are "stored", ignoring synchrotron radiation, than the lower energy muons. Including synchrotron radiation, the time until they hit material and lose sufficient energy so that they strike a detector is also greater. Table 1 gives the characteristics of such electrons vs. the *dp/p* of the parent.

Table 1. Characteristics of decay electrons which are "stored" except for energy loss from synchrotron radiation.

	***dp/p* = -0.25%**	***dp/p* = +0.25%**
***N/N*(*E*>1.8 *GeV*)**	2×10^{-5}	10^{-4}
Av. turns "stored"	3	6

The systematic effect is then:

$$\frac{d\omega}{\omega} \approx < \frac{\Delta t}{\gamma\tau} \frac{d\gamma}{\gamma} > \approx \frac{\left(10^{-4}\right)\left(6 \times 0.15\mu s\right)\left(2.5 \times 10^{-3}\right)}{64\mu s} \approx 3 \times 10^{-9} \qquad (3)$$

This is for the very highest energy electrons, which have long time delays, but there aren't very many of them. Next someone has to do a simulation for the copious electrons above threshold, but below the energies discussed above.

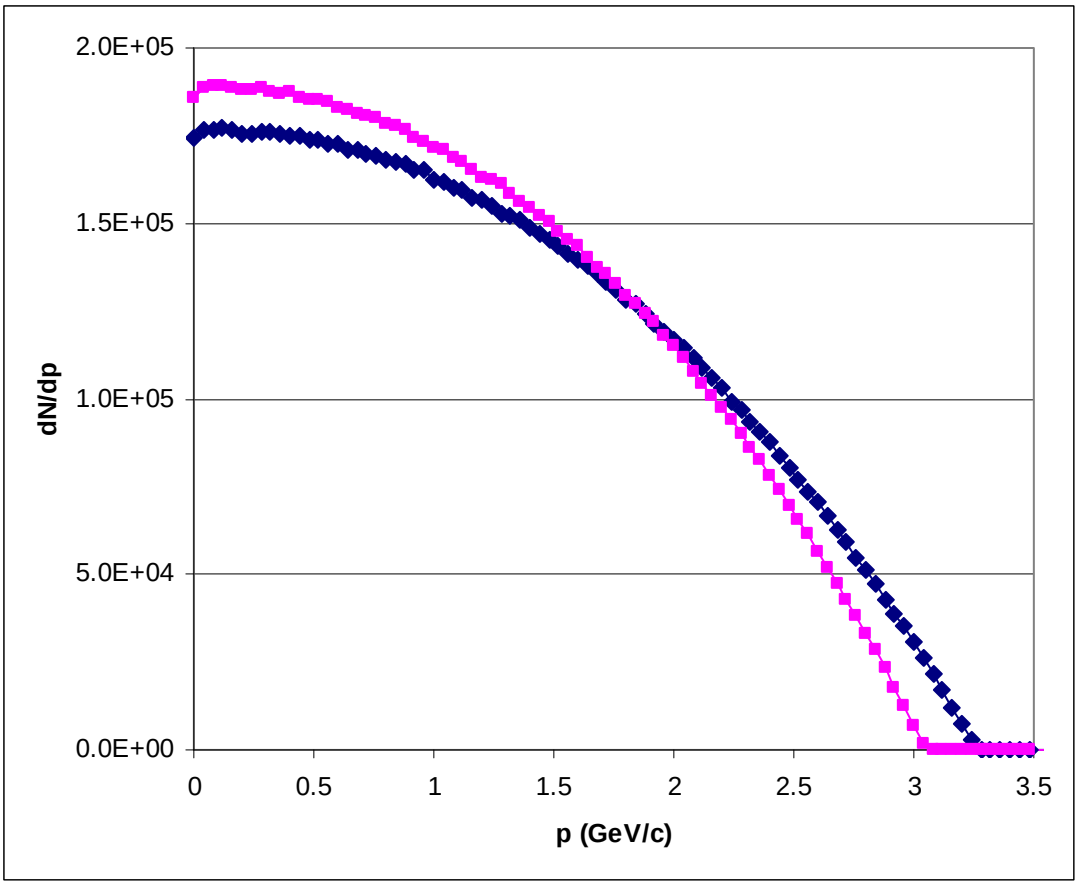


Fig. 1. Decay electron *dN/dp* for muons with 3% higher and lower momentum than the magic momentum to make the effect more visible.

Also, Eric suggested a brilliant idea: increase the magnetic field at a known rate to cancel the first three differential decay systematic effects. However, I realized that we have thick Aluminum vacuum chamber walls above and below the muon storage region, and the eddy currents would not allow the magnetic field to increase over the muon lifetime.

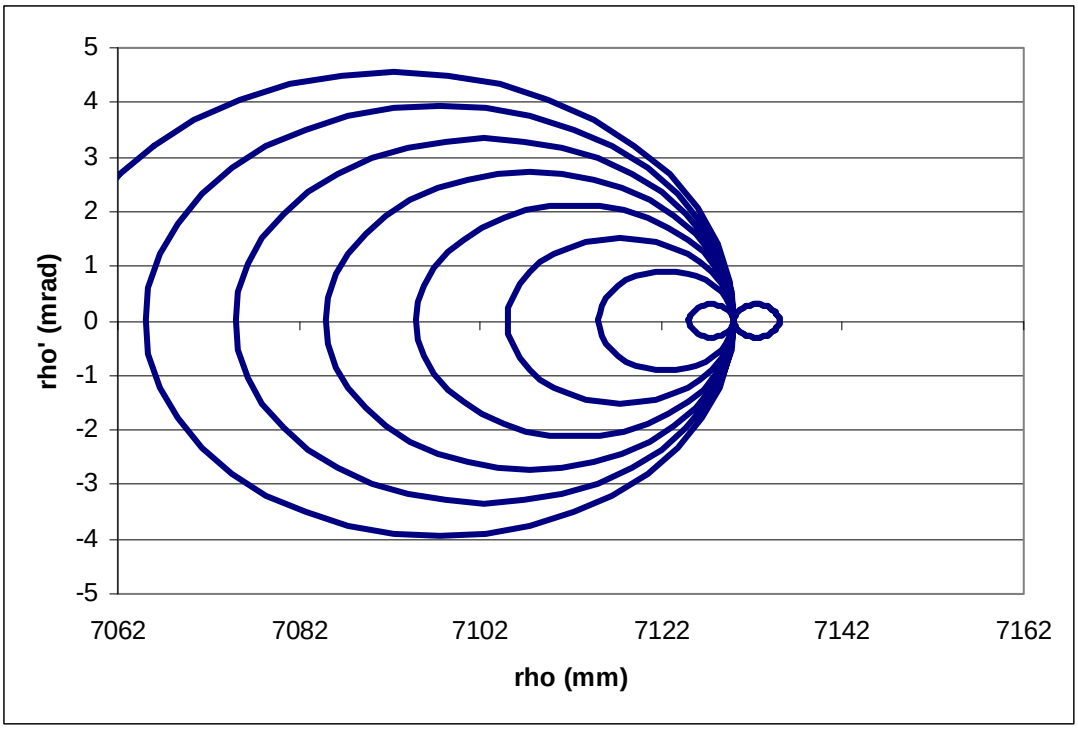


Fig 2a. Muon on it's equilibrium orbit with dp/p = 0.25% with decay electron at the end point, lower by 0.06%, 0.12%, etc. The electron betatron oscillations are shown (ellipses in ρ, ρ' phase space). The kicker and quad electrodes are at ρ = 7062*mm*.

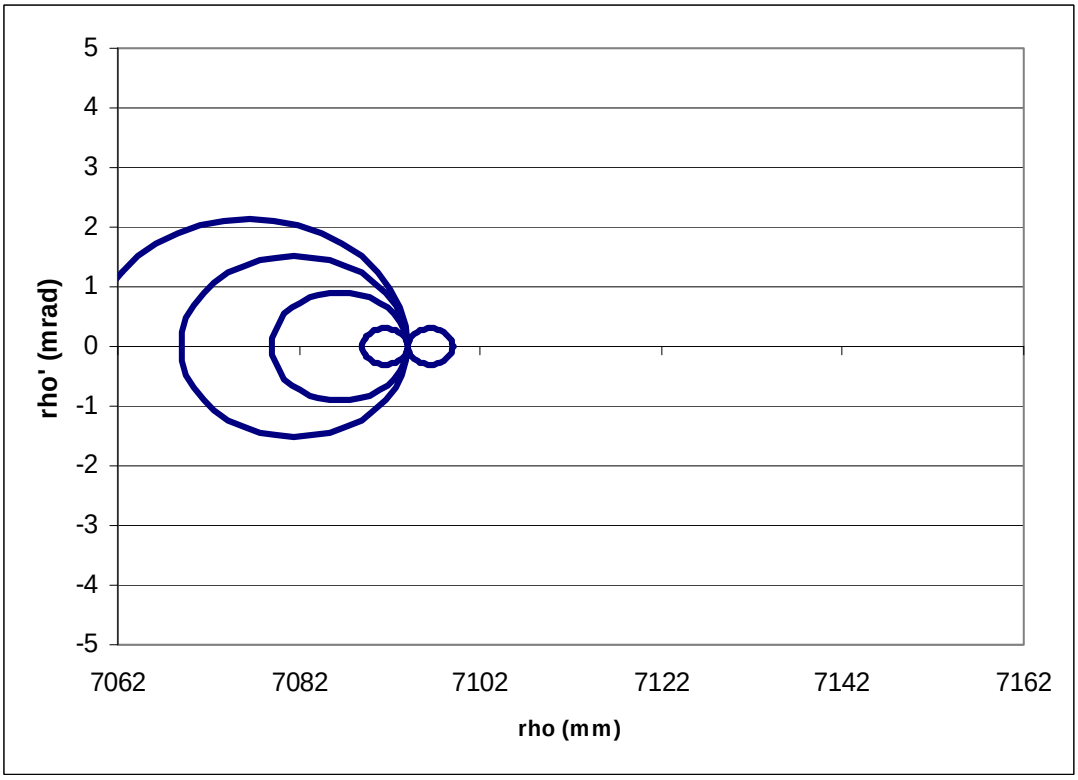


Fig. 2b. Muon on it's equilibrium orbit with dp/p = -0.25% with decay electron at the end point, lower by 0.06%, 0.12%, etc. The electron betatron oscillations are shown.

V. Conclusions.

This note gives our present understanding of the differential decay corrections for E821. Eric thought of a new one: the higher energy muons give higher energy electrons, and these take longer to reach the detectors. I did a simple calculation of the decay electrons which are "stored", except for the energy they lose due to synchrotron radiation. These give the largest time delay, but there aren't very many of them. This differential decay systematic error seems to be small enough (0.003*ppm*). We need someone to do the simulation for the copious electrons above threshold, but below the energies discussed above, or just do it for all decay electrons with synchrotron radiation turned on in GEANT.

References

1. Bennett et al., Phys. Rev D73, 072003 (2006).
2. W. Morse, *The E821 Kicker Field*, g-2 note 22, Dec., 2008.
3. E. Efstathiadis et al., Nucl. Inst. Meth. A496, 8 (2003).
4. W. Morse, g-2 Note 340 (1999).