DEPENDENCE OF ω_a ON ENERGY AND QUAD MISALIGNMENT

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1. Energy Dependence

The anomalous precession frequency is

$$\vec{\omega}_a = -\frac{q}{m}\left[a_\mu \vec{B} - \left(a_\mu - \frac{1}{\gamma^2-1}\right)\frac{\vec{\beta}\times\vec{E}}{c}\right] \tag{1}$$

Energy dependence of $\vec{\omega}_a$

$$\begin{aligned} d\vec{\omega}_a &= -\frac{q}{m} d(\gamma^2-1)^{-1}\frac{\vec{\beta}\times\vec{E}}{c} \\ \frac{d\vec{\omega}_a}{d\gamma} &= \frac{q}{m}\frac{2\gamma}{(\gamma^2-1)^2}\frac{\vec{\beta}\times\vec{E}}{c} \\ \Delta\omega_a &= \frac{q}{m}\frac{2\gamma\Delta\gamma}{(\gamma^2-1)^2}\frac{\vec{\beta}\times\vec{E}}{c} \end{aligned}$$

$$\begin{aligned} \frac{\Delta\omega_a}{\omega_a} &\sim \frac{E/c}{a_\mu B}\frac{2\Delta\gamma}{\gamma^2-1}\frac{\gamma}{\gamma^2}\Delta\gamma \\ &\approx \frac{E/c}{B}\frac{2(\Delta\gamma)}{\gamma} \end{aligned}$$

The electric field in the quadrupole $E = kx$, where for quad voltage of 31kV, $k = 25 \times 10^6 \mathrm{V/m}^2$. The displacement $x = \frac{\Delta\gamma}{\gamma}\eta$ and $\eta \sim 8\mathrm{m}$. Note that the electric field changes sign with change in sign of energy offset. Then

$$\begin{aligned} \frac{\Delta\omega_a}{\omega_a} &\sim 2\frac{k\eta}{Bc}\left(\frac{\Delta\gamma}{\gamma}\right)^2 \\ &= 2\frac{(25\times10^6)(8)}{(1.45)(3\times10^8)}\left(\frac{\Delta\gamma}{\gamma}\right)^2 \\ &= 0.92\left(\frac{\Delta\gamma}{\gamma}\right)^2 \end{aligned}$$

An energy offset of 0.001 corresponds to $\frac{\Delta\omega}{\omega} \sim 0.92$ ppm

2. Quad offset

Suppose the quadrupoles are uniformly displaced by Δr so that the axis of the quadrupoles is at $R_{new} = R + \Delta r$ wheren R is the magic radius. Then the closed orbit for the magic momentum muon will no longer correspond to the magic radius. The momentum of the muon that is centered in the quads will be $p_{cent} = p_0(1 + \Delta r/R)$ where p_0 is the magic momentum. The closed orbit of the muon at the magic radius will be displaced by $-\Delta r$ from the orbit centered in the quads, corresponding to an energy offset of $\delta_o = -\delta r/\eta$. The momentum of the muon at the magic radius, (and with the "magic" revolution frequency) The muon at the magic radius will therefore have momentum

$$p = (1 + \Delta r/R - \Delta r/\eta)p_0 = (1 + \Delta r(\frac{1}{R} - \frac{1}{\eta}))p_0 = (1 + \Delta r\frac{(\eta - R)}{\eta R})p_0$$

and the fractional momentum discrepancy of the muon on the magic radius will be

$$\frac{\Delta\gamma}{\gamma} \sim \frac{\Delta r(\eta - R)}{\eta R}.$$

The electric field is $E = kx = k\Delta r$. Then

$$\frac{\Delta\omega_a}{\omega_a} \sim \frac{k\Delta r}{Bc}\frac{2\Delta\gamma}{\gamma} \tag{2}$$

$$= \frac{2k}{Bc}\frac{(\eta - R)}{\eta R}(\Delta r)^2 \tag{3}$$

If $\Delta r = 1$mm, then

$$p \sim p_0(1 + \frac{0.001}{56}) \sim p_0(1 + 0.02\%)$$

and

$$\frac{\Delta\omega}{\omega_a} \sim 2\frac{(25 \times 10^6)(8 - 7)}{1.45(3 \times 10^8)(7 \cdot 8)}(10^{-3})^2 \sim \frac{50}{243} \times 10^{-8} = 2 \times 10^{-9}$$

Note that the error in ω_a scales with the square of the quad displacement. If the quads were displaced by 5 mm, $(\Delta\omega)/\omega \sim 50$ ppb. That is, if we do not know that the quads are misaligned, and assume that the magic revolution frequency corresponds to the magic momentum, the error in the measurement of ω is very small as long as the displacement of the quads is less than 2-3 mm.

3. Quad displacement and energy offset

In section 1 we found that if the quads are aligned perfectly, then $\Delta\omega_a/\omega_a$ scales as the square of the fractional energy offset, and in particular a part in 1000 energy offset corresponds to 1 ppm change in ω_a. Now again suppose that the quads are uniformly displaced radially by Δr, what is the effect on precession frequency for a muon with momentum with revolution frequency $\Delta f/f = \Delta r/R$? Again the momenum of the muon with magic frequency will be

$$p = (1 + \Delta r\frac{(\eta - R)}{\eta R})p_0$$

The momentum of the muon with frequency $\frac{f+\Delta f}{f}$ will be $p(1+\frac{\Delta r_o}{\eta})$

$$= (1+\Delta r\frac{(\eta-R)}{\eta R}+\frac{\Delta r_o}{\eta})p_0 = (1+\frac{\Delta r_o-\Delta r}{\eta}+\frac{\Delta r}{R})$$

where $\Delta r_0/R = \Delta f/f$. The displacement of the closed orbit from the quad axis is $x = \Delta r_o + \Delta r$. Then Equation 2 gives

$$\begin{aligned}\frac{\Delta\omega_a}{\omega_a} &\sim \frac{k(\Delta r+\Delta r_0)}{Bc}\frac{2\Delta\gamma}{\gamma}\\ &\sim \frac{k(\Delta r+\Delta r_0)}{Bc}2\left(\frac{\Delta r_0-\Delta r}{\eta}+\frac{\Delta r}{R}\right)\end{aligned}$$

Suppose the quads are displaced radially by $\Delta r = 1$ mm. A fractional frequency offset of 0.001 corresponds to $\Delta r_0 = 7$ mm. Then

$$\frac{\Delta\omega_a}{\omega_a} \sim 2\frac{(25\times10^6)(0.007)}{1.45(3\times10^8)}(0.001) = 8\times10^-7$$