Ideal Ring Equations of Motion and Betatron Oscillation

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Notes on the equations of motion in our ring. Derivations of betatron frequencies and equilibrium radii. Assumption is ideal quads and uniform B field, because why not start there? Includes derivation of tune as function of energy. Right now notation need some cleaning up. When δ appears alone, it means $\frac{\delta p}{p_0}$. When it appears in front of a variable, e.g. $\delta\rho$, it stands for a small deviation in that variable from its equilibrium value.

1 Basic Equations of Motion: Cyclotron Frequency and Field Index

We'll do it in cylindrical coordinates, z is towards the ceiling. To start off:

$$\begin{aligned}\boldsymbol{x} &= \rho\hat{\rho} + z\hat{z},\ \dot{\hat{\rho}} = \dot{\theta}\hat{\theta},\ \dot{\hat{\theta}} = -\dot{\theta}\hat{\rho}\\ \dot{\boldsymbol{x}} &= \dot{\rho}\hat{\rho} + \rho\dot{\theta}\hat{\theta} + \dot{z}\hat{z}\\ \ddot{\boldsymbol{x}} &= (\ddot{\rho} - \rho\dot{\theta}^2)\hat{\rho} + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\hat{\theta} + \ddot{z}\hat{z}\end{aligned} \tag{1}$$

Now the forces, set c and e to 1:

$$\boldsymbol{F_B} = \dot{\boldsymbol{x}} \times B_0\hat{z} \tag{2}$$

$$\boldsymbol{F_E} = -kz\hat{z} + k(\rho - \rho_0)\hat{\rho} \tag{3}$$

Relativistic Newton's second law:

$$\begin{aligned}\frac{d\boldsymbol{p}}{dt} &= \boldsymbol{F},\ \boldsymbol{p} = m\gamma\dot{\boldsymbol{x}}\\ \ddot{\boldsymbol{x}} &= \frac{\boldsymbol{F}}{m\gamma}\\ \ddot{\boldsymbol{x}} &= \left[\frac{\rho\dot{\theta}B_0}{m\gamma} + \frac{k(\rho-\rho_0)}{m\gamma}\right]\hat{\rho} - \frac{\dot{\rho}B_0}{m\gamma}\hat{\theta} - \frac{kz}{m\gamma}\hat{z}\end{aligned} \tag{4}$$

This yields three differential equations of motion:

$$\ddot{\rho} - \rho\dot{\theta}^2 = \frac{\rho\dot{\theta}B_0}{m\gamma} + \frac{k(\rho-\rho_0)}{m\gamma}, \tag{5}$$

$$2\dot{\rho}\dot{\theta} + \rho\ddot{\theta} = -\frac{\dot{\rho}B_0}{m\gamma}, \tag{6}$$

and

$$\ddot{z} = -\frac{kz}{m\gamma}. \tag{7}$$

The last one is friendly. In the limit of uniform circular motion, $\dot{\theta} = -\omega_c$ and we can say:

$$\frac{d^2z}{d\theta^2} = -\frac{kz}{m\gamma\omega_c^2}$$

$$\boxed{\frac{d^2z}{d\theta^2} = -nz} \tag{8}$$

Staying in our uniform circular motion limit and looking at a design momentum particle, equation 5 setting $\ddot{\rho} = 0$ and $\rho = \rho_0$ gives us the well known identity $\dot{\theta} = -\frac{B_0}{m\gamma} = \omega_c$. Plugging this along with $\omega_c = v/R_0$ into $n = \frac{k}{m\gamma\omega_c^2}$ gives us

$$\boxed{n = \frac{kR_0}{vB_0}} \tag{9}$$

which is the equation that appears in the E821 final report. n is called the field index.

2 Horizontal Oscillations for Magic Muons

Now that we've got a handle on the basics, let's look at what happens for small oscillations around the equilibrium orbit for a magic muon. I'll define the perturbations to the ideal orbit as:

$$\theta = -\frac{B_0}{m\gamma}t + \delta\theta \tag{10}$$

$$\rho = \rho_0 + \delta\rho \tag{11}$$

Assuming perturbations are small, everything will be kept to only linear order in any small parameter. Assume the quad fields are so weak compared to the muon energy that γ can be treated as constant. We'll need the following quantity:

$$\dot{\theta} = -\frac{B_0}{m\gamma} + \dot{\delta\theta} \tag{12}$$

$$\dot{\theta}^2 = \frac{B_0^2}{m^2\gamma^2} - 2\frac{B_0\dot{\delta\theta}}{m\gamma} + \mathcal{O}(\delta\theta^2) \tag{13}$$

Plugging into equation 5, the zeroth order terms cancel as they already satisfy the equation and we are left with:

$$\ddot{\delta\rho} = -\frac{B_0\rho_0\dot{\delta\theta}}{m\gamma} + \frac{k\delta\rho}{m\gamma} + \mathcal{O}(\delta^2) \tag{14}$$

To eliminate $\dot{\delta\theta}$, we can use equation 6:

$$\begin{aligned} -2\dot{\delta\rho}\frac{B_0}{m\gamma} + \rho_0\ddot{\delta\theta} &= -\dot{\delta\rho}\frac{B_0}{m\gamma} \\ \ddot{\delta\theta} &= \dot{\delta\rho}\frac{B_0}{m\gamma\rho_0} \\ \dot{\delta\theta} &= \delta\rho\frac{B_0}{m\gamma\rho_0} \end{aligned} \tag{15}$$

Putting it all together gives:

$$\ddot{\delta\rho} = -\left(\frac{B_0^2}{m^2\gamma^2} - \frac{k}{m\gamma}\right)\delta\rho \tag{16}$$

This will give periodic behavior, as expected. Since the amplitude of oscillation is very small, we can effectively replace $\dot{\theta}$ with $\langle\dot{\theta}\rangle$ which will be again $-\frac{B_0}{m\gamma}$. Adopting the local coordinate system in which $x \equiv \delta\rho$:

$$\boxed{\frac{d^2x}{d\theta^2} = -(1-n)x} \tag{17}$$

3 Non-magic Muons

Now to examine the case in which the muon is off the magic momentum but in uniform circular motion. In otherwords, I am looking for the equilibrium radius as a function of momentum for muons close to the design momentum. I'll write $\rho = \rho_0 + x_{eq}$ and keep everything to first order in small parameters. Some quantities we'll need to plug into equation 5:

$$\begin{aligned}\dot{\theta} &= -\frac{\beta}{\rho} \\ &= -\frac{\beta}{\rho_0}\left(1 - \frac{x_{eq}}{\rho_0}\right) + \mathcal{O}(x_{eq}^2) \qquad (18)\end{aligned}$$

$$\dot{\theta}^2 = \frac{\beta^2}{\rho_0^2}\left(1 - \frac{2x_{eq}}{\rho_0}\right) \qquad (19)$$

It will also be useful to make the following substitution:

$$\frac{1}{m\gamma} = \frac{\beta}{p_0 + \delta p} \sim \frac{\beta}{p_0}\left(1 - \frac{\delta p}{p_0^2}\right) \qquad (20)$$

Equation 5 now tells us:

$$-(\rho_0 + x_{eq})\frac{\beta^2}{\rho_0^2}\left(1 - \frac{2x_{eq}}{\rho_0}\right) = -(\rho_0 + x_{eq})\frac{\beta}{\rho_0}\left(1 - \frac{x_{eq}}{\rho_0}\right) B_0 \frac{\beta}{p_0}\left(1 - \frac{\delta p}{p_0}\right) + \frac{k x_{eq}}{m\gamma}$$

$$-\frac{\beta^2}{\rho_0} - x_{eq}\frac{\beta^2}{\rho_0^2} + \frac{2\beta^2 x_{eq}}{\rho_0^2} = -\frac{\beta^2 B_0}{p_0} - \frac{x_{eq}\beta^2 B_0}{\rho_0 p_0} + \frac{\beta^2 x_{eq} B_0}{\rho_0 p_0} + \frac{\beta^2 B_0 \delta p}{p_0^2} + \frac{k x_{eq}}{m\gamma}$$

The zeroth order terms must cancel again, and they do since $-\beta^2\left(\frac{1}{\rho_0} - \frac{B_0}{p_0}\right) = -\beta^2\left(\frac{1}{\rho_0} - \frac{\omega_0}{\beta}\right) = 0$. More cancellation occurs and we are left with:

$$\frac{x_{eq}}{\rho_0^2} = \frac{B_0 \delta p}{p_0^2} + \frac{k x_{eq}}{\beta^2 m\gamma}$$

We can make the substitution $\frac{B_0}{p_0} = \frac{\omega_0}{\beta} = \frac{1}{\rho_0}$ to get:

$$\frac{x_{eq}}{\rho_0} = \frac{\delta p}{p_0} + \rho_0 \frac{k x_{eq}}{\beta^2 m\gamma}$$

$$x_{eq} = \rho_0 \frac{\delta p}{p_0\left(1 - \frac{\rho_0^2 k}{\beta^2 m\gamma}\right)}$$

Using $\frac{\rho_0^2}{\beta^2} = \frac{1}{\omega_0^2}$ and the shorthand $\delta = \frac{\delta p}{p_0}$ allows us to write:

$$\boxed{x_{eq} = \rho_0 \frac{\delta}{1 - n}} \qquad (21)$$

And therefore, using $\rho_{eq}\omega_c = \rho_0\omega_0$, which is true to $\mathcal{O}(\delta/\gamma_0^2)$:

$$\boxed{\omega_c(\delta) = \omega_0\left(1 - \frac{\delta}{1 - n}\right)} \qquad (22)$$

The exact expression for x_{eq} in the highly relativistic limit is also not difficult to obtain and is the solution to the equation

$$\frac{-1}{1 + x_{eq}/\rho_0} = -\frac{\gamma_0}{\gamma} + n\frac{x_{eq}\gamma_0}{\rho_0\gamma} \qquad (23)$$

This is a messy expression, though, and the linear approximation should be sufficient.

4 Oscillations for Non-magic Muons

How does the betatron frequency change with δ?

4.1 Vertical

In the highly relativistic limit, we can say $\frac{\delta\gamma}{\gamma_0} = \frac{\delta p}{p_0}$ with errors $\mathcal{O}(\frac{1}{\gamma_0^2})$. Using this approximation and previous results we can make the following manipulations:

$$\begin{aligned}\frac{d^2z}{dt^2} &= -\frac{kz}{m\gamma} \\ &= -\frac{k}{m\gamma_0}\left(1 - \frac{\delta\gamma}{\gamma_0}\right) z \\ &= -\frac{k}{m\gamma_0}(1-\delta)z\end{aligned}$$

To find the change in effective tune, we need to look at $\frac{d^2z}{d\theta^2}$, which will require the use of equation 22:

$$\begin{aligned}\frac{d^2z}{d\theta^2} &= -\frac{k}{\omega_c^2 m\gamma_0}(1-\delta)z \\ \frac{d^2z}{d\theta^2} &= -\frac{k}{\omega_0^2\left(1 - \frac{\delta}{1-n}\right)^2 m\gamma_0}(1-\delta)z \\ \frac{d^2z}{d\theta^2} &= -\frac{k}{\omega_0^2 m\gamma_0^2}\left(1 + \frac{2\delta}{1-n}\right)(1-\delta)z \\ \frac{d^2z}{d\theta^2} &= -n\left(1 + \frac{1+n}{1-n}\delta\right) z \end{aligned} \tag{24}$$

The tune, ν_y, is then:

$$\nu_y = \sqrt{n\left(1 + \frac{1+n}{1-n}\delta\right)}$$

$$\boxed{\nu_y = \sqrt{n} + \frac{1+n}{2-2n}\sqrt{n}\delta} \tag{25}$$

This equation reveals the first order *chromaticity*, which is defined by the relation

$$\nu_y(\delta) = \nu_y + \nu_y'\delta, \tag{26}$$

where ν_y' is the y chromaticity. So, we have:

$$\boxed{\nu_y' = \frac{1+n}{2-2n}\sqrt{n}} \tag{27}$$

4.2 Horizontal

We'll proceed as in section 2 with

$$\theta = -\omega_c(\delta)t + \delta\theta \tag{28}$$

$$\rho = \rho_{eq}(\delta) + \delta\rho \tag{29}$$

Except now ω_c and ρ_{eq} are functions of δ. I now realize this notation is awful because δ is unrelated to $\delta\rho$ and $\delta\theta$. I'll just emphasize that in writing for now, and fix the notation later. Plugging into equation 5:

$$\ddot{\delta\rho} - \rho_{eq}\omega_c^2 + 2\rho_{eq}\omega_c\dot{\delta\theta} - \omega_c^2\delta\rho = -\rho_{eq}\omega_c\frac{B_0}{m\gamma} - \delta\rho\omega_c\frac{B_0}{m\gamma} + \rho_{eq}\dot{\delta\theta}\frac{B_0}{m\gamma} + \frac{k(\rho_{eq}-\rho_0)}{m\gamma} + \frac{k(\delta\rho)}{m\gamma}$$

We know the terms that are not proportional to $\delta\theta$ or $\delta\rho$ will cancel because we picked ρ_{eq} for that to be true in section 3. This leaves us with:

$$\ddot{\delta\rho} + 2\rho_{eq}\omega_c\dot{\delta\theta} - \omega_c^2\delta\rho = -\delta\rho\omega_c\frac{B_0}{m\gamma} + \rho_{eq}\dot{\delta\theta}\frac{B_0}{m\gamma} + \frac{k(\delta\rho)}{m\gamma}$$

There isn't quite as simple as before because $\frac{B_0}{m\gamma}$ is no longer equal to ω_c, but rather $\frac{B_0}{m\gamma} = \omega_0(1-\delta)$. Combining this with equation 22, we get $\frac{B_0}{m\gamma} = \omega_c\left(1+\frac{n}{1-n}\delta\right)$. Using this:

$$\ddot{\delta\rho} + 2\rho_{eq}\omega_c\dot{\delta\theta} - \omega_c^2\delta\rho = -\delta\rho\omega_c^2\left(1+\frac{n}{1-n}\delta\right) + \rho_{eq}\dot{\delta\theta}\omega_c\left(1+\frac{n}{1-n}\delta\right) + \frac{k}{m\gamma}\delta\rho$$

$$\ddot{\delta\rho} = -\left(\rho_{eq}\omega_c - \rho_{eq}\omega_c\frac{n}{1-n}\delta\right)\dot{\delta\theta} - \left(\omega_c^2\frac{n}{1-n}\delta - \frac{k}{m\gamma}\right)\delta\rho$$

Again, we need equation 6 to get rid of $\dot{\delta\theta}$:

$$-2\dot{\delta\rho}\omega_c + \rho_{eq}\ddot{\delta\theta} = -\dot{\delta\rho}\omega_c\left(1+\frac{n}{1-n}\delta\right)$$

$$\ddot{\delta\theta} = \left(\frac{\omega_c}{\rho_{eq}} - \frac{\omega_c}{\rho_{eq}}\frac{n}{1-n}\delta\right)\dot{\delta\rho}$$

$$\dot{\delta\theta} = \frac{\omega_c}{\rho_{eq}}\left(1-\frac{n}{1-n}\delta\right)\delta\rho$$

Plugging this back in above:

$$\ddot{\delta\rho} = -\omega_c^2\left(1-\frac{n}{1-n}\delta\right)\left(1-\frac{n}{1-n}\delta\right)\delta\rho - \left(\omega_c^2\frac{n}{1-n}\delta - \frac{k}{m\gamma}\right)\delta\rho$$

$$\ddot{\delta\rho} = -\left[\omega_c^2\left(1-\frac{n}{1-n}\delta\right) - \frac{k}{m\gamma}\right]\delta\rho$$

Now let's move into $\frac{d^2\delta\rho}{d\theta^2}$. We can also use the same manipulation from section 4.1.

$$\frac{d^2\delta\rho}{d\theta^2} = -\left(1-\frac{n}{1-n}\delta - \frac{k}{m\gamma\omega_c^2}\right)\delta\rho$$

$$\frac{d^2\delta\rho}{d\theta^2} = -\left[1-\frac{n}{1-n}\delta - n\left(1+\frac{1+n}{1-n}\delta\right)\right]\delta\rho$$

$$\frac{d^2\delta\rho}{d\theta^2} = -\left[1-n\left(1+\frac{2+n}{1-n}\delta\right)\right]\delta\rho$$

Therefore:

$$\nu_x(\delta) = \sqrt{1-n-\frac{2n+n^2}{1-n}\delta}$$

$$= \sqrt{1-n} - \frac{2n+n^2}{2(1-n)^{3/2}}\delta$$

$$\boxed{\nu_x' = -\frac{2n+n^2}{2(1-n)^{3/2}}} \tag{30}$$

5 Verification

I verified these equations using a simple GEANT4 simulation with ideal quads at $n = 0.18$ and $\gamma_0 = 29.3$, see Fig. 1.

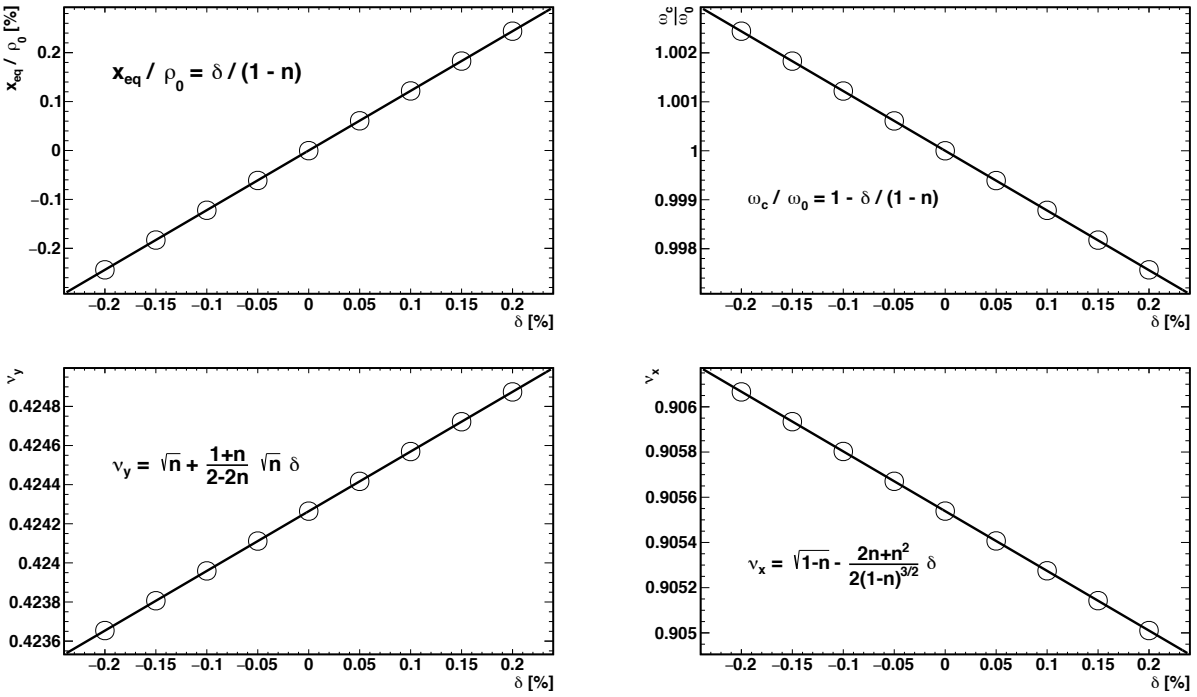

Figure 1: Open circles show results from basic GEANT4 simulation using the classical RK4 integrator. The overlaid lines are first order expressions derived in this note, not fits.