1 On taking the Fourier integral at a finite upper bound

As $t \to \infty$, the signal appears to be the sum of the average muon current $\langle I \rangle = eN_\mu/T$ and an oscillation at the design frequency. At $t \to \infty$ the beam has totally decohered and the signal is DC. The fourier transform will only have the zero frequency. What do you mean by the 'design' frequency? The oscillation at a single frequency is due to the frequencies comprising $S(t)$ undergoing phase mixing, eventually yielding the design frequency as the dominant frequency. I don't understand that last sentence In the preceding Fourier integrals, the upper bound was $t = +\infty$, but in reality we must instead use an acceptable large time t_m. t_m must be a time at which $S(t)$ is exhibiting the asymptotic behavior just described (namely when the beam has decohered and the signal is DC. In that case we get a close approximation to the frequency spectrum, since the frequency content of $S(t)$ does not change much after t_m (the signal keeps oscillating at the same frequency after t_m). In Figures 1 and 2 we Fourier analyze the FRS of a beam with 0.05% Gaussian momentum spread and no initial longitudinal spread. As t_m increases we see that we start to recover the full frequency distribution. Why is the differential in the height of the peak between $t_m = \infty$ and t_m finite greater when $t_m = 350$ vs when $t_m = 100$?

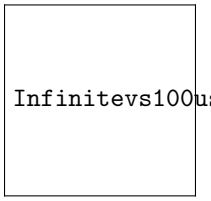

Figure 1: Comparison of spectra obtained using $t_m = 100\mu$s and ∞.

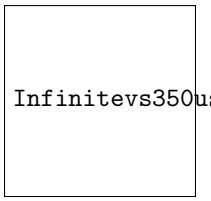

Figure 2: Comparison of spectra obtained using $t_m = 350\mu$s and ∞. Using $t_m = 350\mu$s yields a Gaussian distribution with spread identical to that of the actual distribution, but with amplitude off by factor

2 On the Orlov Delta

To extract a suitable approximation of $\Delta(\omega)$, two considerations need to be taken into account: the presence of frequencies in $\hat{S}_1(\omega)$ outside the aperture region, and the frequency content of $\hat{S}_1(\omega)$ in comparison to that of $\hat{S}(\omega)$. As a reminder

$$\hat{S}_1(\omega) = \sqrt{\frac{2}{\pi}} \int_{t_s}^{\infty} S_1(t) \cos\omega(t - t_0) dt$$

and

$$\hat{S}(\omega) = \sqrt{\frac{2}{\pi}} \int_{t_0}^{\infty} S(t) \cos\omega(t - t_0) dt$$

where $t_s > t_0$ and $S_1(t) = S(t)$ on $[t_s, \infty)$ and vanishes on (t_0, t_s).

2.1 First Consideration

The FRS arises due to the periodic detector hits of the muons, with each muon orbiting within the aperture of the vacuum chamber thus defining the allowed frequencies and their harmonics. $S(t)$ is time reversal symmetric about t_0, and its extension exhibits a form of periodicity in the appearance of peaks occurring at

regular intervals, namely, the revolution period of the muon orbits. $S_1(t)$ on (t_s, ∞) is identically zero for earlier times $(t < t - s)$. The fourier transform of $S_1(t)$ necessarily includes frequencies other than revolution harmonics.

2.2 Second Consideration

$\Delta(\omega)$ will include exactly those non harmonic frequencies that appear in $\hat{S}_1(\omega)$ but with opposite phase so that there is perfect cancellation as only revolution harmonics appear in the sum $\hat{S}(\omega)$. In the limit that $t_s - t_0 \to \ll 2\pi/\omega_0$, where ω_0 is the revolution frequency, $S(t) \to$ constant $= S_0$. Then

$$\hat{S}_0(\omega \sim \omega_0) = \sqrt{\frac{2}{\pi}} S_0 \frac{\sin \omega(t_s - t_0)}{\omega} \to \sqrt{\frac{2}{\pi}} S_0 (t_s - t_0)$$

What if $t_s - t_0)$ includes the first 3 periods. And the first three periods of $S(t)$ are

$$S(t) = \sum_{n=1}^{3} \delta(t - n\tau),$$

Then

$$\Delta(\omega) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{3} \cos(\omega(n\tau - t_0) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{3} \cos(2\pi n \frac{\omega}{\omega_0} - \omega t_0)$$

As n increases the peaks about $\omega \sim n\omega_0$ become increasingly narrow. If n is small, the peaks are very broad. As long as $t_s - t_0$ includes only a few revolution periods, $\Delta(\omega)$ is very nearly uniform in the vicinity of ω_0.

Consider a function $f(t)$ and its Fourier transform $\hat{f}(\omega)$. Then the Fourier transform of $f(at)$ is $\frac{1}{|a|}\hat{f}(\frac{\omega}{a})$. If a is large, then $f(at)$ appears to be squeezed and $\frac{1}{|a|}\hat{f}(\frac{\omega}{a})$ appears to be stretched. If a is small, the opposite occurs. So for large a, the standard deviation of $\frac{1}{|a|}\hat{f}(\frac{\omega}{a})$ is large while for small a the standard deviation of $\frac{1}{|a|}\hat{f}(\frac{\omega}{a})$ is small. This shows that small time windows require more frequencies (resulting in a greater standard deviation for the Fourier transform), and large time windows need less frequencies (resulting in smaller standard deviation for the Fourier transform).

A more quantitative statement of this fact is the Heisenberg uncertainty principle for Fourier transforms. It states that given a function $f(t)$ and its Fourier transform $\hat{f}(\omega)$, their standard deviations satisfy the inequality

$$\sigma(f(t))\sigma(\hat{f}(\omega)) = \left(\int_{-\infty}^{\infty} (t - \bar{t})^2 |f(t)|^2 dt\right)\left(\int_{-\infty}^{\infty} (\omega - \bar{\omega})^2 |\hat{f}(\omega)|^2 d\omega\right) \geq \frac{1}{4\pi}$$

Suppose $f(t)$ is observed on a finite interval (t_0, t_s), with $f(t) \geq 0$. Let t_{max} denote the point in (t_0, t_s) where $(t - \bar{t})^2|f(t)|^2$ is a maximum. Then

$$\sigma(f(t)) = \int_{t_0}^{t_s} (t - \bar{t})^2 |f(t)|^2 dt \leq (t_{max} - \bar{t})^2 |f(t_{max})|^2 (t_s - t_0)$$

As t_s approaches t_0 (the time window gets smaller), $\sigma(f(t))$ becomes smaller, and so $\sigma(\hat{f}(\omega))$ increases according to the uncertainty principle (more frequencies are needed to fit inside the small time window). This means that in the vicinity of $\bar{\omega}$, $\hat{f}(\omega)$ becomes more spread out and so appears flatter, taking on the appearance of a Fourier transform of a Dirac delta function.

In our case, we have $\Delta(\omega)$ instead of $\hat{f}(\omega)$ (Let $s(t)$ denote the FRS on (t_0, t_s), so that $\hat{s}(\omega) = \Delta(\omega)$. Messy notation, I know). Furthermore, we have the constraint that $\hat{S}(\omega) = \hat{S}_1(\omega) + \Delta(\omega)$. Therefore if the frequency content of $\Delta(\omega)$ increases (decreases), then the frequency content of $\hat{S}_1(\omega)$ increases (decreases) albeit with opposite phase. For concreteness, let $S_1(t)$ be contained in (t_0, t_m), where t_m is an appropriate large time (as discussed in section 1 of this note). Then $\sigma(S_1(t)) \leq (t_{max} - \bar{t})^2 |S_1(t_{max})|^2 (t_m - t_s)$, while $\sigma(s(t)) \leq (t'_{max} - \bar{t}')^2 |s(t'_{max})|^2 (t_s - t_0)$. Therefore the standard deviations of $S_1(t)$ and $s(t)$ are inversely related to one another: as $t_s \to t_0$, $\sigma(s(t))$ decreases while $\sigma(S_1(t))$ increases, and vice versa when t_s increases. In turn, when $t_s \to t_0$, $\sigma(\Delta(\omega))$ increases while $\sigma(\hat{S}_1(\omega))$ decreases, and vice versa when t_s increases.

What does this all mean for us? We've seen that in the plots of the frequency spectra, when t_s is on the order of several microseconds, and t_m is appropriately selected, $\Delta(\omega)$ appears to be flat in the vicinity of the vacuum chamber, and thus has a large standard deviation, corresponding to the fact that $s(t)$ is observed in a small time window. Based on our discussion in the previous two paragraphs, this is as it should be. $\hat{S}_1(\omega)$ and $\hat{S}(\omega)$ have much smaller standard deviations, as they should, and in fact their standard deviations appear to be almost the same (see Figure 3). This seems very likely given that the bounds on the standard deviations of $S_1(t)$ and $S(t)$ are $(t_{max}-\bar{t})^2|S_1(t_{max})|^2(t_m-t_s)$ and $(t''_{max}-\bar{t}'')^2|S(t''_{max})|^2(t_m-t_0)$, which are nearly the same since $t_m - t_0 \approx t_m - t_s$ for small enough $t_s - t_0$. By the uncertainty principle the standard deviations of their corresponding Fourier transforms must then be similar. This then means that $\hat{S}_1(\omega)$ and $\hat{S}(\omega)$ differ by nearly a constant within the aperture region (since $\hat{S} - \hat{S}_1 = \Delta$ and Δ is nearly constant in the aperture region). If we drop the frequencies of $\hat{S}_1(\omega)$ outside the aperture region (since they are not physical in the sense discussed in the earlier subsection), then

$$\hat{S}(\omega) - \hat{S}_1(\omega) = \text{const.}$$

Therefore the inverse Fourier transform of $\hat{S}(\omega) - \hat{S}_1(\omega)$ is $S(t) - S_1(t) = \text{const.}\delta(t - t_0)$. What this means is that if we use $\hat{S}_1(\omega)$ for our approximation for $\Delta(\omega)$, we will get nearly the same answer if we used $\hat{S}(\omega)$, which is what we are aiming for.

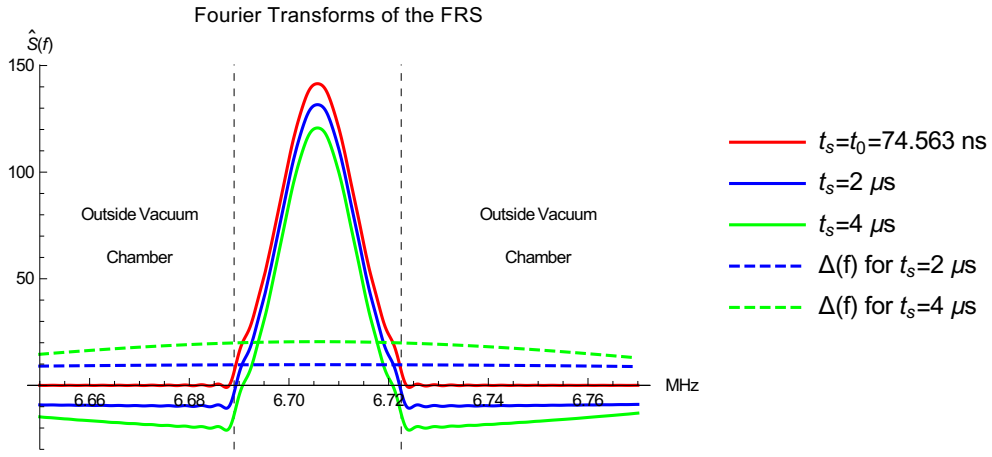

Figure 3: Frequency spectra for the case of a Gaussian energy offset distribution and no longitudinal distribution. Spectra for different t_s are displayed, as well the corresponding $\Delta(f)$ functions. Up to 2000 turns.