Contribution to polarization tilt angle from EDM and radial B-field

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The polarization $\mathbf{S}$ of the muon evolves along its trajectory according to

$$\frac{d}{ds}\mathbf{S} = \left\{ \frac{1+\mathbf{r}\cdot\mathbf{g})}{c\beta_z}(\mathbf{\Omega}_{BMT} + \Omega_{EDM}) - (\mathbf{g}\times\hat{\mathbf{z}} \right\} \times \mathbf{S}$$

where

$$\mathbf{\Omega}_{BMT} = -\frac{q}{mc}\left[\left(\frac{1}{\gamma}+a\right)c\mathbf{B} - \frac{a\gamma c}{1+\gamma}(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta} - \left(a+\frac{1}{1+\gamma}\right)\boldsymbol{\beta}\times\mathbf{E}\right]$$

$$\mathbf{\Omega}_{EDM} = -\frac{q\eta}{2mc}\left[\mathbf{E} - \frac{\gamma}{1+\gamma}(\boldsymbol{\beta}\cdot\mathbf{E})\boldsymbol{\beta} + c\boldsymbol{\beta}\times\mathbf{B}\right]$$

The contribution to $\mathbf{\Omega}_{BMT}$ due to the radial component of the magnetic field is

$$\mathbf{\Omega}^{rad}_{BMT} = -\frac{q}{mc}\left(\frac{1}{\gamma}+a\right)cB_{rad}\hat{\mathbf{x}} \tag{1}$$

Meanwhile, an EDM contributes

$$\mathbf{\Omega}_{EMD} = -\frac{q\eta}{2mc}c\boldsymbol{\beta}\times\mathbf{B}$$

Now according to the Lorentz force law, the force on the particle is given by

$$F = q(\boldsymbol{\beta}\times\mathbf{B})$$

The direction of the force is radially inward (by definition), that is, in the $-\hat{\mathbf{x}}$ direction. Therefore the contributon from the EDM is

$$\mathbf{\Omega}^{rad}_{EMD} = -\frac{q\eta}{2mc}c(-\beta B_z)\hat{\mathbf{x}} \tag{2}$$

Sure enough the contribution to the rate of change of the polarization in the radial direction,($\frac{d}{ds}\mathbf{S}$) due a magnetic field in the +x direction (radially outward) (1) and an EDM with $\eta > 0$ (2) are in opposite directions.

I. LONGITUDINAL FIELD

Suppose there is a longitudinal field along the magic circumference of the form

$$B_l(\phi) = B_{l0}\sin(m\phi) \tag{3}$$

and $m = 1$. The general solution to the Laplacian in cylindrical coordinates is

$$\psi = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho)(\sinh(k_{mn}z)(A_{mn}\sin m\phi + B_{mn}\cos m\phi) + \cosh(k_{mn}z_{(}C_{mn}\sin m\phi + D_{mn}\sin m\phi))$$

Then $\mathbf{B} = \nabla\psi \rightarrow$

$$B_z = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho)k_{mn}(\cosh(k_{mn}z)(A_m\sin m\phi + B_m\cos m\phi) - \sinh(k_{mn}z)(C_m\sin m\phi + D_m\cos m\phi))$$

$$B_\rho = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{\partial J_m}{\partial\rho}(k_{mn}\rho)(\sinh(k_{mn}z)(A_m\sin m\phi + B_m\cos m\phi) + \cosh(k_{mn}z)(C_m\sin m\phi + D_m\cos m\phi))$$

$$B_\phi = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{J_m(k_{mn}\rho)}{\rho}m(\sinh(k_{mn}z)(A_m\cos m\phi - B_m\sin m\phi) + \cosh(k_{mn}z)(C_m\cos m\phi - D_m\sin m\phi))$$

For the form of longitudinal field in Equation 3, $m = 1$. Then

$$\begin{aligned}
B_z &= \sum_{n=1}^{\infty} J_1(k_{1n}\rho)k_{1n}(\cosh(k_{1n}z)(A_{1n}\sin\phi + B_{1n}\cos\phi) - \sinh(k_{1n}z)(C_{1n}\sin\phi + D_{1n}\cos\phi)) \\
B_\rho &= \sum_{n=1}^{\infty} \frac{\partial J_1}{\partial\rho}(k_{1n}\rho)(\sinh(k_{1n}z)(A_{1n}\sin\phi + B_{1n}\cos\phi) + \cosh(k_{1n}z)(C_{1n}\sin\phi + D_{1n}\cos\phi)) \\
B_\phi &= \sum_{n=1}^{\infty} \frac{J_1(k_{1n}\rho)}{\rho}(\sinh(k_{1n}z)(A_{1n}\cos\phi - B_{1n}\sin\phi) + \cosh(k_{1n}z)(C_{1n}\cos\phi - D_{1n}\sin\phi))
\end{aligned}$$

Consider boundary conditions such that when $z = 0$ (where $\rho_0 = \rho_{magic}$), $B_z = 0$ and when $\rho = \rho_{bl}$, $B_z = 0$ for all z. Then $A_{1n} = B_{1n} = 0$, and $k_{11}\rho_0 = x_{11}$ (the first zero of J_1).

$$\begin{aligned}
B_z &= J_1(k_{11}\rho)k_{11}(\sinh(k_{11}z)(C_{11}\sin\phi + D_{11}\cos\phi) \\
B_\rho &= \frac{\partial J_1}{\partial\rho}(k_{11}\rho)(\cosh(k_{11}z)(C_{11}\sin\phi + D_{11}\cos\phi)) \\
B_\phi &= \frac{J_1(k_{11}\rho)}{\rho}\cosh(k_{11}z)(C_{11}\cos\phi - D_{11}\sin\phi))
\end{aligned}$$

where $k_{11} = x_{11}/\rho_{bl}$ And let's chose the phase so that $C_{1n} = 0$

$$\begin{aligned}
B_z &= J_1(k_{11}\rho)k_{11}(\sinh(k_{11}z)(D_{11}\cos\phi) \\
B_\rho &= \frac{\partial J_1}{\partial\rho}(k_{11}\rho)(\cosh(k_{11}z)(D_{11}\cos\phi)) \\
B_\phi &= -\frac{J_1(k_{11}\rho)}{\rho}\cosh(k_{11}z)(D_{11}\sin\phi)
\end{aligned}$$

Evidently

$$B_{l0} = -\frac{J_1(k_{11}\rho_0)}{\rho_0}D_{11}$$

and

$$D_{11} = -B_{l0}\frac{\rho_0}{J_1(k_{11}\rho_0)}$$

Then

$$\begin{aligned}
B_\rho(z=0) &= D_1\frac{\partial J_1}{\partial\rho}(k_{11}\rho)\cos\phi \\
B_z(z=0) &= 0
\end{aligned}$$

We really should sum over all n.

In general

$$\begin{aligned}
B_z &= \sum_n J_1(k_{1n}\rho)k_{1n}(\sinh(k_{1n}z)(C_{1n}\sin\phi + D_{1n}\cos\phi) \\
B_\rho &= \sum_n \frac{\partial J_1}{\partial\rho}(k_{1n}\rho)(\cosh(k_{1n}z)(C_{1n}\sin\phi + D_{1n}\cos\phi)) \\
B_\phi &= \sum_n \frac{J_1(k_{1n}\rho)}{\rho}\cosh(k_{1n}z)(C_{1n}\cos\phi - D_{1n}\sin\phi))
\end{aligned}$$

where $k_{1n} = x_{1n}/\rho_{bl}$ And let's chose the phase so that $C_{1n} = 0$

$$\begin{aligned}
B_z &= \sum_n J_1(k_{1n}\rho)k_{1n}(\sinh(k_{1n}z)(D_{1n}\cos\phi) \\
B_\rho &= \sum_n \frac{\partial J_1}{\partial\rho}(k_{1n}\rho)(\cosh(k_{1n}z)(D_{1n}\cos\phi)) \\
B_\phi &= -\sum_n \frac{J_1(k_{1n}\rho)}{\rho}\cosh(k_{1n}z)(D_{1n}\sin\phi)
\end{aligned}$$

If only a single n contributes

$$B_{l0} = -\frac{J_1(k_{1n}\rho_0)}{\rho_0} D_{1n}$$

and

$$D_{1n} = -B_{l0}\frac{\rho_0}{J_1(k_{1n}\rho_0)}$$

Then

$$\begin{aligned}
&B_\rho(z=0) = D_1\frac{\partial J_1}{\partial \rho}(k_{1n}\rho)\cos\phi \\
&B_z(z=0) = 0
\end{aligned}$$