1 Optical Cooling

Consider optical stochastic cooling using dependence on transit time through the bypass to couple transverse and longitudinal phase space in the pickup to phase in the kicker. The packet emits radiation in the pickup undulator that will arrive in the kicker with some relative phase $\phi = k\Delta s$, where k is the wavenumber of the characteristic undulator radiation and $\Delta s = s - s_0$ is the change in path length through the bypass. The interaction of the packet with the radiation in the kicker shifts its energy by

$$\Delta p/p = \xi \sin(\phi) = \xi \sin(k\Delta s). \tag{1}$$

In order to effect cooling, the phase is necessarily correlated with the phase space coordinate of the packet in the kicker, $\phi(\vec{x}_p)$. That is, the phase depends $\vec{x}_p$. The linear dependence of Δs on $\vec{x}_p$ is written

$$\Delta s = M_{51}x_p + M_{52}x'_p + M_{56}z'_p \tag{2}$$

where M is the 6X6 transfer matrix from the center of the pickup undulator to the center of the kicker. Since $x = x_\beta + x_e$ and $x' = x'_\beta + x'_e$ equation 2 becomes

$$\begin{aligned} \Delta s &= M_{51}(x_\beta + x_e) + M_{52}(x'_\beta + e'_e) + M_{56}z'_p \\ \Delta s &= M_{51}x_\beta + +M_{52}x'_\beta + (M_{51}\eta + M_{52}\eta' + M_{56})z'_p \end{aligned} \tag{3}$$

Next we write phase space coordinates at the pickup in terms of betatron amplitude and phase

$$\begin{aligned} x_{p\beta} &= a\sqrt{\beta_p}\cos\theta \\ x'_{p\beta} &= \frac{1}{2}\frac{a\beta'_p}{\sqrt{\beta_p}}\cos\theta - \frac{a}{\sqrt{\beta_p}}\sin\theta && (4) \\ &= -\frac{a}{\sqrt{\beta_p}}(\alpha_p\cos\theta + \sin\theta) && (5) \end{aligned}$$

and likewise at the kicker for future reference

$$\begin{aligned} x_{k\beta} &= a\sqrt{\beta_k}\cos(\theta+\phi) && (6) \\ x'_{k\beta} &= -\frac{a}{\sqrt{\beta_k}}(\alpha_k(\cos(\theta+\phi) + \sin(\theta+\phi)) && (7) \end{aligned}$$

Then

$$\begin{aligned} \Delta s &= a(M_{51}\sqrt{\beta_p}\cos\theta) - M_{52}\frac{(\alpha_p\cos\theta + \sin\theta)}{\sqrt{\beta_p}}) - a_z(M_{51}\eta + M_{52}\eta' + M_{56})\frac{(\alpha_p\cos\theta_z + \sin\theta_z)}{\sqrt{\beta_z}}) && (8) \\ \Delta s &= A_x\sin(\theta_x + \theta_{x0}) + A_z\sin(\theta_z + \theta_{z0}) && (9) \end{aligned}$$

where

$$\begin{aligned} A_x &= a_x\left[M_{51}^2\beta_x + M_{52}^2\gamma_x - 2M_{51}M_{52}\alpha_x\right]^{1/2} && (10) \\ A_z &= a_z(M_{51}\eta + M_{52}\eta' + M_{56})\gamma_z && (11) \end{aligned}$$

2 Cooling

The cooling is quantified as the change in the invariant amplitude due to interaction of packet with radiation in the kicker undulator. At the kicker $\Delta x_{k\beta} = -\eta_k\Delta p/p$ and $\Delta x'_{k\beta} = -\eta'_k\Delta p/p$. And $\Delta z_k = 0, \Delta z'_k = \Delta p/p$. If $x = a_x\sqrt{\beta_x}\cos\phi_x$, or $z = a_x\sqrt{\beta_z}\cos\phi_z$ then the amplitude

$$a_x^2 = \beta x'^2 + \gamma x^2 + 2\alpha x x'$$

$$\begin{aligned}
\Delta\epsilon_x = 2a_x\Delta a_x &= -2(\Delta p/p)(\beta_x x'_{k\beta}\eta'_x + \gamma_x x_{k\beta}\eta_x + \alpha_x(x_{k\beta}\eta'_k + x'_{k\beta}\eta_k)) && (12)\\
\Delta\epsilon_x &= -2(\Delta p/p)((\gamma_x\eta_x + \alpha_x\eta'_k)\sqrt{\beta_x}\cos\theta - (\beta_x\eta'_x + \alpha_x\eta_k)(\frac{\alpha_x\cos\theta + \sin\theta}{\sqrt{\beta_x}})\\
\Delta\epsilon_x &= -2(\Delta p/p)((\gamma_x\eta_x + \alpha_x\eta'_k - \alpha_x\eta' - \frac{\alpha_x^2\eta_k}{\beta})\sqrt{\beta_x}\cos\theta - (\beta_x\eta'_x + \alpha_x\eta_k)(\frac{\sin\theta}{\sqrt{\beta_x}})\\
\Delta\epsilon_x &= -2(\Delta p/p)(\frac{\eta}{\sqrt{\beta_x}}\cos\theta - (\beta_x\eta'_x + \alpha_x\eta_k)\frac{\sin\theta}{\sqrt{\beta_x}})\\
&= -2(\Delta p/p)E_x\sin(\theta_{xk} + \theta_{xc}) && (13)
\end{aligned}$$

where

$$\begin{aligned}
E_x &= (\frac{\eta^2}{\beta} + \frac{\beta^2(\eta')^2 + \alpha^2\eta^2 + 2\alpha\beta\eta'\eta}{\beta})^{1/2}\\
&= (\eta^2\gamma + \beta{\eta'}^2 + 2\alpha\eta'\eta)^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
\Delta\epsilon_z = 2a_z\Delta a_z &= 2(\Delta p/p)(\beta_z z'_k + \alpha_z z) && (14)\\
&= 2(\Delta p/p)a_z(-\sqrt{\beta_z}(\alpha_z\cos\theta_z + \sin\theta_z) + \alpha_z\sqrt{\beta}\cos\theta_z)\\
&= -2(\Delta p/p)a_z\sqrt{\beta_z}\sin\theta_z\\
&= -2(\Delta p/p)E_z\sin\theta_{zk}\\
& && (15)
\end{aligned}$$

Combining equations 1 and 12 we find

$$\begin{aligned}
\Delta\epsilon_x &= -2(\xi\sin(k\Delta s))\left((\beta_x x'_{k\beta}\eta'_x + \gamma_x x_{k\beta}\eta_x + \alpha_x(x_{k\beta}\eta'_k + x'_{k\beta}\eta_k)) + (\beta_z z'_k + \alpha_z z)\right) && (16)\\
&= -2\xi\sin(k\Delta s))(E_x\sin(\theta_{xk} + \theta_{xc})) && (17)\\
&= -2\xi\sin(k(A_x\sin(\theta_{xp} + \theta_{xt}) + A_z\sin(\theta_{zp} + \theta_{zt}))(E_x\sin(\theta_{xk} + \theta_{xc})) && (18)
\end{aligned}$$

Now let's average over all betatron phases

$$\begin{aligned}
\int_0^{2\pi}\Delta\epsilon_x d\theta_x d\theta_z &= -2\xi\int\sin(k(A_x\sin(\theta_{xp} + \theta_{xt}) + A_z\sin(\theta_{zp} + \theta_{zt}))(E_x\sin(\theta_{xk} + \theta_{xc}))d\theta_x d\theta_z && (19)\\
&= -2\xi\int\sin(k(A_x\sin(\theta_x) + A_z\sin(\theta_z + \theta_{zt}))(E_x\sin(\theta_x + \theta_0 + \theta_{xc} - \theta_{xt}))d\theta_x d\theta_z && (20)
\end{aligned}$$

where we use the fact that thet betatron phase advance from pickup to kicker is θ_0, that is $\theta_{xk} = \theta_{xp} + \theta_0$ Then

$$\begin{aligned}
\langle\Delta\epsilon_x\rangle &= -2\xi E_x\int[\sin(k(A_x\sin(\theta_x))\cos(kA_z\sin(\theta_z + \theta_{zt})) + && (21)\\
&\quad \cos(k(A_x\sin(\theta_x))\sin(kA_z\sin(\theta_z + \theta_{zt}))](\sin(\theta_x + \theta_0 + \theta_{xc} - \theta_{xt}))d\theta_x d\theta_z && (22)
\end{aligned}$$

We will use the Bessel integral

$$J_n(x) = \frac{1}{\pi}\int_0^\pi\cos(n\tau - x\sin(\tau))d\tau = \frac{1}{\pi}\int_0^\pi(\cos(n\tau)\cos(x\sin\tau) + \sin(n\tau)\sin(x\sin\tau))\,d\tau$$

To be checked but it looks like, using the Bessel integral

$$\langle\Delta\epsilon_x\rangle = -\frac{\xi}{\pi}E_x J_1(kA_x)J_0(kA_z) \qquad (23)$$

In that limit where $k\Delta s \ll \pi/2$, and with substitution of equation 2 into 16 we have

$$\Delta\epsilon_x = -2(\xi k(M_{51}x_p + M_{52}x'_p + M_{56}z'_p)(\beta_x x'_{k\beta}\eta'_x + \gamma_x x_{k\beta}\eta_x + \alpha_x(x_{k\beta}\eta'_k + x'_{k\beta}\eta_k)) \quad (24)$$

We compute the average change in the emittance $\langle\Delta\epsilon_x\rangle$ where the average is over betatron phase. Substituting Equations 4-7 into 24 and averaging over betatron phase (see Appendix for details)

$$\langle\Delta\epsilon_x\rangle = -2\pi\xi k\frac{a^2}{2}(M_{51}\left(-\sqrt{\beta_p\beta_k}\sin\phi\eta'_k + \sqrt{\frac{\beta_p}{\beta_k}}\eta_k(\cos\phi - \alpha_k\sin\phi)\right) \quad (25)$$

$$+M_{52}\left(\sqrt{\frac{\beta_k}{\beta_p}}\eta'_k(\cos\phi + \alpha_p\sin\phi) + \sqrt{\frac{1}{\beta_k\beta_p}}\eta_k(\sin\phi(1+\alpha_k\alpha_p) + \cos\phi(\alpha_k - \alpha_p))\right) \quad (26)$$

Consider a couple of special cases. If the phase advance ϕ from pickup to kicker is $\phi = \pi$ then

$$\langle\Delta\epsilon_x\rangle = -2\pi\xi k\frac{a^2}{2}(M_{51}\left(-\sqrt{\frac{\beta_p}{\beta_k}}\eta_k\right) + M_{52}\left(-\sqrt{\frac{\beta_k}{\beta_p}}\eta'_k - \sqrt{\frac{1}{\beta_k\beta_p}}\eta_k\cos\phi(\alpha_k - \alpha_p))\right)$$

and if the optics are symmetric so that $\beta_k = \beta_p, \alpha_k = -\alpha_p$, $\eta_k = \eta_p$,$\eta'_k = -\eta'_p$ then

$$\langle\Delta\epsilon_x\rangle = 2\pi\xi k\frac{a^2}{2}(M_{51}\eta + M_{52}\left(\eta'_k + \frac{\eta}{\beta}\cos\phi(2\alpha_k)\right)$$

3 Sample Lengthening

As noted above, cooling requires that the change in path length be less than the optical wavelength, $\Delta s < \lambda$. Substitution of Equations 4 and 5 into the expression for the change in path length 3

The average change in path length is of course $\langle\Delta s\rangle = 0$. The mean square change in path length is

$$\langle(\Delta s)^2\rangle = \frac{\pi}{2}\left(a^2(M_{51}^2\beta_p + M_{52}^2\gamma - 2M_{51}M_{52}\alpha) + a_z^2(M_{51}\eta + M_{52}\eta' + M_{56})^2\gamma_z)\right) \quad (27)$$

a^2 and a_z^2 are the horizontal and longitudinal emittances respectively. Particles with amplitudes within one standard deviation of the emittance will be cooled if $\sqrt{\langle(\Delta s)^2)\rangle} < \lambda$.

4 Damping

The matrix that maps from kicker to pickup is M_{kp} and from pickup to kicker M_{pk}. At the kicker

$$\Delta\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Delta p/p \end{pmatrix} = M_e M_l \vec{x}_p = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \xi k & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_{51} & M_{52} & 0 & M_{56} \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{x}_p$$

where $\vec{x}_p$ is the phase space vector in the pickup. Then the effect of a single turn is

$$\vec{x}_{k,n+1} = M_{pk}M_{kp}\vec{x}_n + \Delta\vec{x} = (M_e M_l + M_{pk})M_{kp}\vec{x}_{k,n} = T\vec{x}_{k,n} \quad (28)$$

The full turn matrix at the kicker is

$$T = \Delta M + M$$

where

$$\Delta M = M_e M_l M_{kp}$$
$$M = M_{pk}M_{kp}$$

Compute the eigenvectors ($\vec{v}_i$) and eigenvalues of M. We know how to do this since we have standard methods for diagonalizing a symplectic matrix. (The eigenvalues are $\lambda_x^\pm = e^{\pm i\mu_x}$ and $\lambda_z^\pm = e^{\pm i\mu_z}$) where μ_x and μ_z are the horizontal and longitudinal tunes.) Then in the limit where ΔM is small, (it clearly scales with $\xi k M_{5j}^{pk}$) the shift in the eigenvalues (tunes) is given by

$$\Delta\lambda_i \sim \vec{v}_i^T(\Delta M)\vec{v}_i$$

An imaginary component will correspond to damping.

4.1 Pickup to Kicker matrix

Next to work out the matrix M_{pk} that maps pickup to kicker. We can write

$$M_{pk} = \begin{pmatrix} A_{pk} & B_{pk} \\ C_{pk} & D_{pk} \end{pmatrix}$$

And

$$C = \begin{pmatrix} M_{51} & M_{52} \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & M_{56} \\ 0 & 1 \end{pmatrix}$$

The symplectic condition requires that

$$\begin{aligned} ASA^T + BSB^T &= S \\ ASC^T + BSD^T &= 0 \\ CSA^T + DSB^T &= 0 \\ CSC^T + DSD^T &= S \end{aligned}$$

from which we can conclude that

$$B = ASC^T(D^T)^{-1}S$$

For simplicity we suppose $\alpha_p = \alpha_k = 0$. Then

$$\begin{aligned} A_{pk} &= \begin{pmatrix} \cos\mu_x & \beta_x\sin\mu_x \\ -\frac{\sin\mu_x}{\beta_x} & \cos\mu_x \end{pmatrix} \\ D_{pk} &= \begin{pmatrix} 1 & M_{56} \\ 0 & 1 \end{pmatrix} \\ B_{pk} &= \begin{pmatrix} \cos\mu_x & \beta_x\sin\mu_x \\ -\frac{\sin\mu_x}{\beta_x} & \cos\mu_x \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} M_{51} & 0 \\ M_{52} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -M_{56} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos\mu_x & \beta_x\sin\mu_x \\ -\frac{\sin\mu_x}{\beta_x} & \cos\mu_x \end{pmatrix}\begin{pmatrix} M_{52} & 0 \\ -M_{51} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -M_{56} \end{pmatrix} = \begin{pmatrix} \cos\mu_x & \beta_x\sin\mu_x \\ -\frac{\sin\mu_x}{\beta_x} & \cos\mu_x \end{pmatrix}\begin{pmatrix} 0 & M_{52} \\ 0 & -M_{51} \end{pmatrix} \quad (29) \end{aligned}$$

where μ_x is the phase advance from pickup to kicker. We assume $\beta_p = \beta_k$. If $\eta_k = \eta_p$ and $\eta_k' = \eta_p' = 0$ then

$$B_{pk} = (I - A_{pk})\begin{pmatrix} 0 & \eta \\ 0 & 0 \end{pmatrix}$$

then from 29

$$\begin{aligned} B_{pk} &= (I - A_{pk})\begin{pmatrix} 0 & \eta \\ 0 & \eta' \end{pmatrix} = A_{pk}\begin{pmatrix} 0 & M_{52} \\ 0 & -M_{51} \end{pmatrix} \\ &\rightarrow A_{pk}^{-1}(I - A_{pk})\begin{pmatrix} 0 & \eta \\ 0 & \eta' \end{pmatrix} = \begin{pmatrix} 0 & M_{52} \\ 0 & -M_{51} \end{pmatrix} \\ &\rightarrow A_{pk}^{-1}(I - A_{pk})\begin{pmatrix} 0 & \eta \\ 0 & \eta' \end{pmatrix} = \begin{pmatrix} 0 & M_{52} \\ 0 & -M_{51} \end{pmatrix} \end{aligned}$$

Evidently M_{5i} and dispersion are dependent and the product of dispersion and M_{5i} in Equation 11

$$\eta M_{51} + \eta' M_{52} = \begin{pmatrix} M_{51} & M_{52} \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} M_{51} & M_{52} \end{pmatrix} A_{pk}^{-1}(I - A_{ft}) \begin{pmatrix} M_{52} \\ -M_{51} \end{pmatrix}$$

Not sure what we learned with the above but at least now I know how to write the full turn at the pickup and the kicker, that is assuming they are the same, and neglecting RF.

$$\begin{aligned}
CSA^T + DSB^T &= 0 \to C = -DSB^T(A^T)^{-1}S \\
C &= -\begin{pmatrix} 1 & M_{56} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \eta & \eta' \end{pmatrix} (A^T)^{-1}S \\
C &= -\begin{pmatrix} 1 & M_{56} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \eta & \eta' \\ 0 & 0 \end{pmatrix} (A^T)^{-1}S \\
C &= -\begin{pmatrix} \eta & \eta' \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\mu & \gamma\sin\mu \\ -\beta\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} \eta & \eta' \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\gamma\sin\mu & \cos\mu \\ -\cos\mu & -\beta\sin\mu \end{pmatrix} \\
&= \begin{pmatrix} \eta\gamma\sin\mu + \eta'\cos\mu & -\eta\cos\mu + \eta'\beta\sin\mu \\ 0 & 0 \end{pmatrix} \\
M &= \begin{pmatrix} A & \begin{pmatrix} 0 & \eta \\ 0 & \eta' \end{pmatrix} \\ \begin{pmatrix} \eta & \eta' \\ 0 & 0 \end{pmatrix} (A^T)^{-1}S & \begin{pmatrix} 1 & M_{56} \\ 0 & 1 \end{pmatrix} \end{pmatrix}
\end{aligned}$$

The coupling matrix

$$\begin{aligned}
m + n^\dagger &= \begin{pmatrix} 0 & \eta \\ 0 & \eta' \end{pmatrix} + -SA^T \begin{pmatrix} 0 & -\eta' \\ 0 & \eta \end{pmatrix} \\
&= \begin{pmatrix} 0 & \eta \\ 0 & \eta' \end{pmatrix} + SA^TS \begin{pmatrix} 0 & \eta \\ 0 & \eta' \end{pmatrix} = (I + A^{-1}) \begin{pmatrix} 0 & \eta \\ 0 & \eta' \end{pmatrix} \\
C &= \frac{m + n^\dagger}{tr(A - D) + |m + n^\dagger|}
\end{aligned}$$

The eigenvectors of the rotation matrix are $\vec{v} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$ with eigenvalues $e^{\pm i\mu}$. It appears that

$$U = V^{-1}MV \to R(\mu_x, \mu_z) = G^{-1}V^{-1}MVG$$

Then the eigenvalues of M are

$$\begin{aligned}
\vec{m}_i &= VG\vec{v}_i \to \Delta\lambda_i = \vec{v}_i^T G^T V^T \Delta M V G \vec{v}_i \\
&= \vec{v}_i^T G^T \begin{pmatrix} \gamma & -(C^\dagger)^T \\ C & \gamma \end{pmatrix} \begin{pmatrix} 0 & 0 \\ M_l & M_r \end{pmatrix} \begin{pmatrix} \gamma & C \\ -C^\dagger & \gamma \end{pmatrix} G\vec{v}_i \\
&= \vec{v}_i^T G^T \begin{pmatrix} \gamma & -(C^\dagger)^T \\ C & \gamma \end{pmatrix} \begin{pmatrix} 0 & 0 \\ M_l\gamma - M_r C^\dagger & M_l C + \gamma M_r \end{pmatrix} G\vec{v}_i
\end{aligned}$$

The eignvectors of the full turn matrix are

$$\vec{v} =$$

5 Generalized kicker parameters

At the kicker $\Delta x_{k\beta} = -\eta_k \Delta p/p$ and $\Delta x'_{k\beta} = -\eta'_k \Delta p/p$.The action

$$a^2 = \beta {x'}^2 + \gamma x^2 + 2\alpha x x'$$

$$2a\Delta a = -2\Delta p/p(\beta x'_{k\beta}\eta'_k + \gamma x_{k\beta}\eta_k + \alpha(x_{k\beta}\eta'_k + x'_{k\beta}\eta_k)) \quad (30)$$

Now if the phase advance from pickup to kicker is 180 degrees, then $x_{k\beta} = -x_{p\beta}$ and $x'_{k\beta} = -x'_{p\beta}$ and

$$2a\Delta a = 2\Delta p/p(\beta x'_{p\beta}\eta'_k + \gamma x_{p\beta}\eta_k + \alpha(x_{p\beta}\eta'_k + x'_{p\beta}\eta_k))$$

$$= 2\Delta p/p\left(\eta'_k(\beta x'_{p\beta} + \alpha x_{p\beta}) + \eta_k(\gamma x_{p\beta} + \alpha x'_{p\beta})\right)$$

$$= 2(\Delta p/p)a\left(\eta'_k(-\sqrt{\beta}\sin\theta) + \eta_k(\frac{\cos\theta - \alpha\sin\theta}{\sqrt{\beta}})\right)$$

6 Cooling

Since $\Delta p/p = \xi\sin(k\Delta s)$ we have that

$$2a\Delta a = 2a\xi\sin(k\Delta s)\left(\eta'_k(-\sqrt{\beta_k}\sin\theta) + \eta_k(\frac{\cos\theta - \alpha_k\sin\theta}{\sqrt{\beta_k}})\right)$$

$$2a\Delta a = 2a\xi\sin\left[ka\left(M_{51}\sqrt{\beta_p}\cos\theta - M_{52}\frac{(\alpha_p\cos\theta + \sin\theta)}{\sqrt{\beta_p}}\right)\right]\left(\eta'_k(-\sqrt{\beta_k}\sin\theta) + \eta_k(\frac{\cos\theta - \alpha_k\sin\theta}{\sqrt{\beta_k}})\right)$$

In the limit where $k\Delta s \ll \pi/2$, we can write that

$$\Delta a = \xi\left[ka\left(M_{51}\sqrt{\beta_p}\cos\theta - M_{52}\frac{(\alpha_p\cos\theta + \sin\theta)}{\sqrt{\beta_p}}\right)\right]\left(\eta'_k(-\sqrt{\beta_k}\sin\theta) + \eta_k(\frac{\cos\theta - \alpha_k\sin\theta}{\sqrt{\beta_k}})\right)$$

$$\langle\Delta a\rangle = -\frac{a}{2}\xi k\left(M_{51}\eta_k\sqrt{\frac{\beta_p}{\beta_k}} + M_{52}\left(\frac{\eta_k(\alpha_p - \alpha_k)}{\sqrt{\beta_p\beta_k}} + \eta'_k\sqrt{\frac{\beta_k}{\beta_p}}\right)\right)$$

If $\alpha_k = -\alpha_p$ and $\beta_k = \beta_p$

$$\langle\Delta a\rangle = -\frac{a}{2}\xi k\left(M_{51}\eta_k + M_{52}\left(2\frac{\eta_k(\alpha_k)}{\beta_p} + \eta'_k\right)\right)$$

7 Longitudinal excitation

While the momentum shift $\Delta p/p$ is designed to damp the transverse motion, it is apparently adding noise to the longitudinal. As long as sychrotron and betatron tunes are not related the average momentum shift will be zero. Not a problem? If M_{56} is finite then

$$\Delta s = (M_{51}\eta + M_{52}\eta' + M_{56})\delta$$

$$\Delta p/p = \xi\sin(k(M_{51}\eta + M_{52}\eta' + M_{56})\delta)$$

and there will be longitudinal cooling if the sign of ξ is chosen appropriately. But this in turn will add uncorrelated noise into the transverse.

8 More general

Suppose the betatron phase advance from pickup to kicker is θ_0 so that

$$\begin{aligned} x_{k\beta} &= a\sqrt{\beta_k}\cos(\phi + \theta_0) \\ x'_{k\beta} &= -\frac{a}{\sqrt{\beta_k}}\left(\alpha_k\cos(\phi + \theta_0) + \sin(\phi + \theta_0)\right) \end{aligned}$$

Since

$$
\begin{aligned}
x_{p\beta} &= a\sqrt{\beta_p}\cos(\phi) \\
x'_{p\beta} &= -\frac{a}{\sqrt{\beta_p}}\left(\alpha_p\cos(\phi)+\sin(\phi)\right)
\end{aligned}
$$

we can write

$$
\begin{aligned}
a\cos\phi &= \frac{x_{p\beta}}{\sqrt{\beta_p}} \\
a\sin\phi &= -\sqrt{\beta_{p\beta}}x'_{p\beta} - \frac{\alpha_p}{\sqrt{\beta_p}}x_{p\beta}
\end{aligned}
$$

Then

$$
\begin{aligned}
x_{k\beta} &= \sqrt{\beta_k}\left(\frac{x_{p\beta}}{\sqrt{\beta_p}}\cos\theta_0 + (\sqrt{\beta_{p\beta}}x'_{p\beta} + \frac{\alpha_p}{\sqrt{\beta_p}}x_{p\beta})\sin\theta_0\right) \\
x'_{k\beta} &= -\frac{1}{\sqrt{\beta_k}}\left(\alpha_k(\frac{x_{p\beta}}{\sqrt{\beta_p}}\cos\theta_0 + (\sqrt{\beta_{p\beta}}x'_{p\beta} + \frac{\alpha_p}{\sqrt{\beta_p}}x_{p\beta})\sin\theta_0) + \frac{x_{p\beta}}{\sqrt{\beta_p}}\sin\theta_0 - (\sqrt{\beta_{p\beta}}x'_{p\beta} + \frac{\alpha_p}{\sqrt{\beta_p}}x_{p\beta})\cos\theta_0\right)
\end{aligned}
$$

Let's write $2a\Delta a$ in terms of $x_{p\beta}, x'_{p\beta}$.

9 Another try

$$
2a\Delta a = -2\xi k(M_{51}x_{p\beta} + M_{52}x'_{p\beta})(\beta x'_{k\beta}\eta'_k + \gamma x_{k\beta}\eta_k + \alpha(x_{k\beta}\eta'_k + x'_{k\beta}\eta_k)) \tag{31}
$$

Then we have terms like

$$
\begin{aligned}
\langle x_p x_k\rangle &= \langle\sqrt{\beta_k}\left(\frac{x_{p\beta}^2}{\sqrt{\beta_p}}\cos\theta_0 + (\sqrt{\beta_{p\beta}}x_p x'_{p\beta} + \frac{\alpha_p}{\sqrt{\beta_p}}x_{p\beta}^2)\sin\theta_0\right)\rangle \\
\langle x_p x_k\rangle &= \frac{a^2}{2}\sqrt{\beta_k}\left(\frac{\beta_p}{\sqrt{\beta_p}}\cos\theta_0 + (-\sqrt{\beta_{p\beta}}\alpha_p + \frac{\alpha_p}{\sqrt{\beta_p}}\beta_p)\sin\theta_0\right) \\
\langle x_p x_k\rangle &= \frac{a^2}{2}\sqrt{\beta_k\beta_p}\left(\cos\theta_0\right)
\end{aligned}
$$

Next

$$
\begin{aligned}
\langle x_p x'_{k\beta}\rangle &= \langle -\frac{a^2}{\sqrt{\beta_k}}\Bigg(\alpha_k(\frac{x_p x_{p\beta}}{\sqrt{\beta_p}}\cos\theta_0 + (\sqrt{\beta_{p\beta}}x_p x'_{p\beta} + \\
&\quad \frac{\alpha_p}{\sqrt{\beta_p}}x_p x_{p\beta})\sin\theta_0) + \frac{x_p x_{p\beta}}{\sqrt{\beta_p}}\sin\theta_0 - (\sqrt{\beta_{p\beta}}x_p x'_{p\beta} + \frac{\alpha_p}{\sqrt{\beta_p}}x_p x_{p\beta})\cos\theta_0\Bigg)\rangle \\
\langle x_p x'_{k\beta}\rangle &= -\frac{1}{2}\frac{a^2}{\sqrt{\beta_k}}\Bigg(\alpha_k(\frac{\beta_p}{\sqrt{\beta_p}}\cos\theta_0 + (\sqrt{\beta_{p\beta}}(-\alpha_p) + \\
&\quad \frac{\alpha_p}{\sqrt{\beta_p}}\beta_p)\sin\theta_0) + \frac{\beta_p}{\sqrt{\beta_p}}\sin\theta_0 - (-\sqrt{\beta_{p\beta}}\alpha_p + \frac{\alpha_p}{\sqrt{\beta_p}}\beta_p)\cos\theta_0\Bigg) \\
\langle x_p x'_{k\beta}\rangle &= -\frac{a^2}{2}\frac{\sqrt{\beta_p}}{\sqrt{\beta_k}}(\alpha_k\cos\theta_0 + \sin\theta_0)
\end{aligned}
$$

Another term

$$
\begin{aligned}
\langle x'_p x_{k\beta}\rangle &= \langle x'_p \sqrt{\beta_k}\left(\frac{x_{p\beta}}{\sqrt{\beta_p}}\cos\theta_0 + (\sqrt{\beta_{p\beta}}x'_{p\beta} + \frac{\alpha_p}{\sqrt{\beta_p}}x_{p\beta})\sin\theta_0\right) \\
\langle x'_p x_{k\beta}\rangle &= \frac{a^2}{2}\sqrt{\beta_k}\left(\frac{-\alpha_p}{\sqrt{\beta_p}}\cos\theta_0 + (\sqrt{\beta_{p\beta}}\gamma_{p\beta} - \frac{\alpha_p^2}{\sqrt{\beta_p}})\sin\theta_0\right) \\
\langle x'_p x_{k\beta}\rangle &= \frac{a^2}{2}\sqrt{\frac{\beta_k}{\beta_p}}(\sin\theta_0 - \alpha_p\cos\theta_0)
\end{aligned}
$$

Finally

$$
\begin{aligned}
\langle x'_p x'_{k\beta}\rangle &= -x'_p\frac{1}{\sqrt{\beta_k}}\left(\alpha_k(\frac{x_{p\beta}}{\sqrt{\beta_p}}\cos\theta_0 + (\sqrt{\beta_{p\beta}}x'_{p\beta} + \frac{\alpha_p}{\sqrt{\beta_p}}x_{p\beta})\sin\theta_0) + \frac{x_{p\beta}}{\sqrt{\beta_p}}\sin\theta_0 - (\sqrt{\beta_{p\beta}}x'_{p\beta} + \frac{\alpha_p}{\sqrt{\beta_p}}x_{p\beta})\cos\theta_0\right) \\
&= -\frac{a^2}{2\sqrt{\beta_k}}\left(-\frac{\alpha_k\alpha_p}{\sqrt{\beta_p}}\cos\theta_0 + \alpha_k(\sqrt{\beta_p}\gamma_p - \frac{\alpha_p^2}{\sqrt{\beta_p}})\sin\theta_0 - \frac{\alpha_p}{\sqrt{\beta_p}}\sin\theta_0 - (\sqrt{\beta_p}\gamma_p - \frac{\alpha_p^2}{\sqrt{\beta_p}})\cos\theta_0\right) \\
&= -\frac{a^2}{2\sqrt{\beta_k}}\left(-\frac{\alpha_k\alpha_p}{\sqrt{\beta_p}}\cos\theta_0 + \frac{\alpha_k}{\sqrt{\beta_p}}\sin\theta_0 - \frac{\alpha_p}{\sqrt{\beta_p}}\sin\theta_0 - (\sqrt{\beta_p}\gamma_p - \frac{\alpha_p^2}{\sqrt{\beta_p}})\cos\theta_0\right) \\
&= -\frac{a^2}{2\sqrt{\beta_k}}\left(-\frac{\alpha_k\alpha_p}{\sqrt{\beta_p}}\cos\theta_0 - \frac{1}{\sqrt{\beta_p}}\cos\theta_0 + \frac{\alpha_k - \alpha_p}{\sqrt{\beta_p}}\sin\theta_0\right) \\
&= -\frac{a^2}{2\sqrt{\beta_k\beta_p}}((-1 - \alpha_k\alpha_p)\cos\theta_0 + (\alpha_k - \alpha_p)\sin\theta_0)
\end{aligned}
$$

Now we can write Equation 31 Step 1

$$
\begin{aligned}
2a\Delta a &= -2\xi k(M_{51}x_{p\beta} + M_{52}x'_{p\beta})(\beta x'_{k\beta}\eta'_k + \gamma x_{k\beta}\eta_k + \alpha(x_{k\beta}\eta'_k + x'_{k\beta}\eta_k)) \\
&= -2\xi k(M_{51}\left(x_p\beta x'_k\eta'_k + \gamma_k\eta_k x_p x_k + \alpha_k(\eta'_k x_p x_k + \eta_k x_p x'_k)\right) + \\
&\quad M_{52}\left(\beta_k\eta'_k x'_p x'_k + \gamma_k\eta_k x'_p x_k + \alpha(\eta'_k x'_p x_k + \eta_k x'_p x'_k)\right))
\end{aligned}
$$

Step 2

$$
\begin{aligned}
&= -2\xi k(M_{51}(\eta'_k\left(\beta_k x_p x'_k + \alpha_k x_p x_k\right) + \eta_k\left(\gamma_k x_p x_k + \alpha_k x_p x'_k\right)) + \\
&\quad M_{52}(\eta'_k\left(\beta_k x'_p x'_k + \alpha x'_p x_k\right) + \eta_k\left(\gamma_k x'_p x_k + \alpha x'_p x'_k\right))
\end{aligned}
$$

Step 3

$$
\begin{aligned}
&= -2\xi k\frac{a^2}{2}(M_{51}\left(-\sqrt{\beta_p\beta_k}.(\alpha_k\cos\theta_0 + \sin\theta_0)\eta'_k + \gamma_k\eta_k\sqrt{\beta_k\beta_p}\cos\theta_0\right. \\
&\quad \left.+\alpha_k(\sqrt{\beta_k\beta_p}\eta'_k\cos\theta_0 - \sqrt{\frac{\beta_p}{\beta_k}}\alpha_k(\alpha_k\cos\theta_0 + \sin\theta_0)\eta_k\right) \\
&\quad +M_{52}\left((1 + \alpha_k\alpha_p)\cos\theta_0 + (\alpha_p - \alpha_k)\sin\theta_0)\sqrt{\frac{\beta_k}{\beta_p}}\eta'_k + \sqrt{\frac{\beta_k}{\beta_p}}(\sin\theta_0 - \alpha_p\cos\theta_0)\gamma_k\eta_k +\right. \\
&\quad \left.\alpha_k(\sqrt{\frac{\beta_k}{\beta_p}}(\sin\theta_0 - \alpha_p\cos\theta_0)\eta'_k + \alpha_k\eta_k\frac{1}{\sqrt{\beta_k\beta_p}}((1 + \alpha_k\alpha_p)\cos\theta_0 + (\alpha_p - \alpha_k)\sin\theta_0)\right)
\end{aligned}
$$

Step 4

$$= \quad -2\xi k\frac{a^2}{2}(M_{51}\left(-\sqrt{\beta_p\beta_k}\sin\theta_0\eta_k' + \sqrt{\frac{\beta_p}{\beta_k}}\eta_k\left(\cos\theta_0 - \alpha_k\sin\theta_0\right)\right)$$

$$+M_{52}\left(\sqrt{\frac{\beta_k}{\beta_p}}\eta_k'(\cos\theta_0 + \alpha_p\sin\theta_0) + \frac{1}{\sqrt{\beta_k\beta_p}}\eta_k\left(\sin\theta_0(1+\alpha_k\alpha_p) + (\alpha_k - \alpha_p)\cos\theta_0\right)\right)$$

Step 5

$$= \quad -2\xi k\frac{a^2}{2}(M_{51}\left(-\sqrt{\beta_p\beta_k}\sin\theta_0\eta_k' + \sqrt{\frac{\beta_p}{\beta_k}}\eta_k(\cos\theta_0 - \alpha_k\sin\theta_0)\right)$$

$$+M_{52}\left(\sqrt{\frac{\beta_k}{\beta_p}}\eta_k'(\cos\theta_0 + \alpha_p\sin\theta_0) + \sqrt{\frac{1}{\beta_k\beta_p}}\eta_k(\sin\theta_0(1+\alpha_k\alpha_p) + \cos\theta_0(\alpha_k - \alpha_p))\right)$$

If we have symmetry

$$= \quad -2\xi k\frac{a^2}{2}(M_{51}\left(-\beta_p\sin\theta_0\eta_k' + \eta_k(\cos\theta_0 - \alpha_k\sin\theta_0) + M_{52}\left(\eta_k'(\cos\theta_0 + \alpha_p\sin\theta_0) + \frac{\eta_k}{\beta}(\sin\theta_0(1-\alpha^2) + 2\alpha_k\cos\theta_0\right)\right.$$

and if $\theta_0 = \pi$

$$= \quad 2\xi k\frac{a^2}{2}(M_{51}\eta_k + M_{52}\eta_k' + 2\frac{\eta_k}{\beta}\alpha_k)$$

And if $\theta_0 = \pi/2$

$$= \quad -2\xi k\frac{a^2}{2}(M_{51}\left(-\beta_p\eta_k' + \eta_k(-\alpha_k) + M_{52}\left(\eta_k'(\alpha_p) + \frac{\eta_k}{\beta}((1-\alpha^2)\right)\right.$$

$$2a\Delta a = -2\Delta p/p(\beta x_{k\beta}'\eta_k' + \gamma x_{k\beta}\eta_k + \alpha_k(x_{k\beta}\eta_k' + x_{k\beta}\eta_k))$$