Multipole Expansion

Separation of variables and multipole expansion are two sides of the same cosmic coin

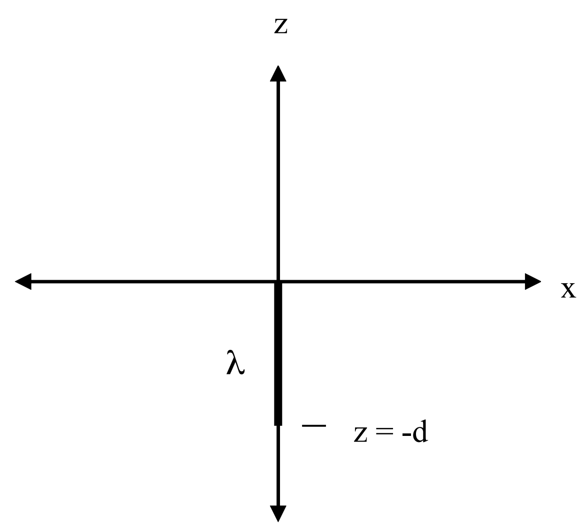
Part 1 – Potential from a Line of Charge

A uniform line charge density λ extends from the origin to the point (0,0,-d).

i. Using the script-r technique from earlier in the course, find an expression for the potential for points on the positive z-axis, V(z).

Recall:

$$V = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho(\bar{r}')d\tau'}{|\mathscr{r}|}$$



Your answer to part i. should have been: $V(z)=\frac{\lambda}{4\pi\varepsilon_o}\ln(1+\frac{d}{z})$. (If you didn't get this, find out where you went wrong.) In the next part, we will be interested in V(z) for *large* z (i.e. small $\frac{d}{z}$). Defining a small parameter $\varepsilon\equiv\frac{d}{z}$, we get $V(z)=\frac{\lambda}{4\pi\varepsilon_o}\ln(1+\varepsilon)$, which can be Taylor expanded in powers of ε.

ii. Find the first two non-zero terms of this Taylor expansion.

Part 2 – Separation of Variables

i. Consider the charge distribution from Part 1. This problem does not have spherical symmetry. Could the potential have the form: $V(r,\theta) = \sum_l (A_l \cdot r^l + \frac{B_l}{r^{l+1}}) \cdot P_l(\cos\theta)$?

If so, in what regions could the solution look like this?

ii. Assuming the potential can have the form $V(r,\theta) = \sum_l (A_l \cdot r^l + \frac{B_l}{r^{l+1}}) \cdot P_l(\cos\theta)$ in the region you specified above, find the two leading non-zero A's and/or B's. Do any terms vanish? Keep in mind that this potential is the same potential you solved by integrating in part 1, so when θ = 0 (the z-axis), the answers must match.

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$
$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$

Part 3 – Multipole expansion

A potential can be expanded into the form:

$$V = \frac{1}{4\pi\varepsilon_o}\left(\frac{"monopole"}{r} + \frac{"dipole"}{r^2} + \frac{"quadrapole"}{r^3} + \ldots\right)$$

i. For this problem, what are the monopole and dipole moments? Do these answers make sense physically?

ii. Which terms would change if the charge distribution were shifted up by d/2, so that it was centered on the origin?

iii. Does your answer to part ii make sense physically? What is the physical significance of the dipole term when there is a net charge?