A ring with charge $Q$ and radius $R$ is rotating about its axis with period $T$.

1. Create an integral expression for the magnetic field caused by this ring everywhere in space. We know that

$$B(r) = \frac{\mu_0 I}{4\pi} \int \frac{dV' \times \hat{r}}{r^2}$$

[Solution: Let’s evaluate $B$ at $r = y\hat{y} + z\hat{z}$. (We might as well assume the point is in the y-z plane and set $x = 0$.) The line element $dV'$ is at $r' = a\cos\theta\hat{x} + a\sin\theta\hat{y}$ Then

$$r = -a\cos\theta\hat{x} + (y - a\sin\theta)\hat{y} + z\hat{z}$$

and

$$r^2 = a^2 + y^2 - 2ya\sin\theta + z^2$$

The line element

$$dV' = a\theta(\sin\theta\hat{x} - \cos\theta\hat{y})$$

and

$$dV' \times r = \begin{pmatrix} \hat{i} \\ ad\theta\sin\theta \\ -ad\theta\cos\theta \end{pmatrix} \times \begin{pmatrix} \hat{j} \\ -a\cos\theta \\ y - a\sin\theta \end{pmatrix} \hat{k}$$

$$= (-z\cos\theta\hat{i} - z\sin\theta\hat{j} + (y\sin\theta - a\cos^2\theta)\hat{k})ad\theta$$

$$= -z(\cos\theta\hat{i} + \sin\theta\hat{j}) + (y\sin\theta - a)\hat{k}ad\theta$$

$$= (-z\hat{s} + (y\sin\theta - a)\hat{k})ad\theta$$

So

$$B = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{(-z(\cos\theta\hat{i} + \sin\theta\hat{j}) + (y\sin\theta - a)\hat{k})ad\theta}{(a^2 + y^2 - 2ya\sin\theta + z^2)^{3/2}}$$

If $r$ is on the z-axis, then

$$B = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{(-z(\cos\theta\hat{i} + \sin\theta\hat{j}) - a\hat{k})ad\theta}{(a^2 + z^2)^{3/2}}$$

$$B = \frac{\mu_0 I}{4\pi} \frac{2\pi a^2\hat{k}}{(a^2 + z^2)^{3/2}}$$
2. Create an integral expression for the magnetic vector potential caused by the ring everywhere in space.

\[ A(r) = \frac{\mu_0}{4\pi} I \int \frac{dV'}{r} \]

Solution:

\[ A(r) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} a d\theta \hat{\phi} \left( \frac{a^2 + y^2 - 2ya\sin \theta + z^2}{(a^2 + y^2 - 2ya\sin \theta + z^2)^{1/2}} \right) \]

On the z-axis

\[ A(r) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} a d\theta \hat{\phi} \left( \frac{a^2 + z^2}{(a^2 + z^2)^{1/2}} \right) \]

3. Develop a power series expansion for the vector potential near the center.

4. Develop a power series expansion for the vector potential for \( r \gg R \).