P3323 Rotating Ring
October 14, 2016

A ring with charge Q and radius R is rotating about its axis with period T.

1. Create an integral expression for the magnetic field caused by this ring everywhere in space. We know that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

[Solution: Let's evaluate $\mathbf{B}$ at $\mathbf{r} = y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. (We might as well assume the point is in the y-z plane and set $x = 0$.) The line element $d\mathbf{l}'$ is at $\mathbf{r}' = a\cos\theta\hat{\mathbf{x}} + a\sin\theta\hat{\mathbf{y}}$ Then

$$\mathscr{r} = -a\cos\theta\hat{\mathbf{x}} + (y - a\sin\theta)\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

and

$$\mathscr{r}^2 = a^2 + y^2 - 2ya\sin\theta + z^2$$

The line element

$$d\mathbf{l}' = ad\theta(\sin\theta\hat{\mathbf{x}} - \cos\theta\hat{\mathbf{y}})$$

and

$$\begin{aligned}
d\mathbf{l}' \times \mathscr{r} &= \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ ad\theta\sin\theta & -ad\theta\cos\theta & 0 \\ -a\cos\theta & (y - a\sin\theta) & z \end{pmatrix} \\
&= (-z\cos\theta\hat{\mathbf{i}} - z\sin\theta\hat{\mathbf{j}} + (\sin\theta(y - a\sin\theta) - a\cos^2\theta)\hat{\mathbf{k}})ad\theta \\
&= -z(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}) + (y\sin\theta - a)\hat{\mathbf{k}})ad\theta \\
&= (-z\hat{\mathbf{s}} + (y\sin\theta - a)\hat{\mathbf{k}})ad\theta
\end{aligned}$$

So

$$\mathbf{B} = \frac{\mu_0}{4\pi} I \int_0^{2\pi} \frac{(-z(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}) + (y\sin\theta - a)\hat{\mathbf{k}})ad\theta}{(a^2 + y^2 - 2ya\sin\theta + z^2)^{3/2}}$$

If $\mathbf{r}$ is on the z-axis, then

$$\mathbf{B} = \frac{\mu_0}{4\pi} I \int_0^{2\pi} \frac{(-z(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}) - a\hat{\mathbf{k}})ad\theta}{(a^2 + z^2)^{3/2}}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} I \frac{2\pi a^2\hat{\mathbf{k}}}{(a^2 + z^2)^{3/2}}]$$

2. Create an integral expression for the magnetic vector potential caused by the ring everywhere in space.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}'}{\mathscr{r}}$$

[Solution:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int_0^{2\pi} \frac{a d\theta \hat{\phi}}{(a^2 + y^2 - 2ya\sin\theta + z^2)^{1/2}}$$

On the z-axis

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} I \int_0^{2\pi} \frac{a d\theta \hat{\phi}}{(a^2 + z^2)^{1/2}} \\ &= \frac{\mu_0}{4\pi} I \frac{2\pi a \hat{\phi}}{(a^2 + z^2)^{1/2}} \end{aligned}$$

3. Develop a power series expansion for the vector potential near the center.

4. Develop a power series expansion for the vector potential for $r \gg R$.