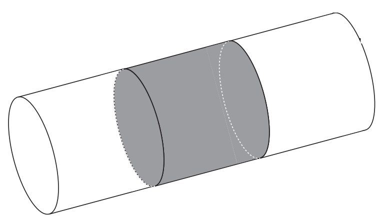
PHYS 3323 - Prelim II

1. **[16 points] Multiple choice. Circle the correct answer.**

   A. A capacitor is charged by a battery with voltage V_0. When fully charged, the stored charge is Q_0 and the field between the plates is E_0. The battery is disconnected and a dielectric slab is inserted between the plates. Which quantity is the same before and after the dielectric is inserted?

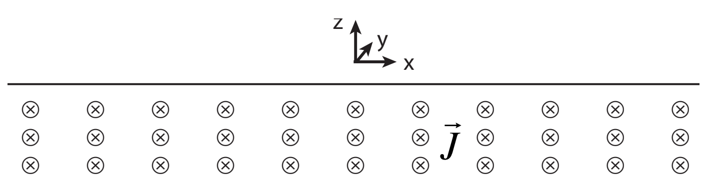
   a. The charge on the plates (originally Q_0)
   b. The potential between the plates (originally V_0)
   c. The field between the plates (originally E_0)
   d. A and B
   e. All of the above

   B. A resistor is spliced into a wire as shown below. Both wire and resistor consist of Ohms law materials, they have conductivities σ_w and σ_r, respectively, that satisfy $\sigma_w > \sigma_r$, and both are cylindrical with the same radius. A uniform, steady state current I flows through the system. Which statement is true?



   a. J and E are larger in the resistor than in the wire
   b. J and E are smaller in the resistor than in the wire
   c. E is the same in both, but J is larger in the resistor
   d. J is the same in both, but E is larger in the resistor
   e. need more geometry information to tell
   f. none of the above

   C. A very large conducting slab has a current density flowing into the page ($\hat{\mathbf{y}}$ direction) as shown. What is the direction of the vector potential above the plate, assuming that $\nabla \cdot \mathbf{A} = 0$ so that our standard derivations apply.



   a. upwards ($+\hat{\mathbf{z}}$)
   b. downwards ($-\hat{\mathbf{z}}$)
   c. right ($+\hat{\mathbf{x}}$)
   d. left ($-\hat{\mathbf{x}}$)
   e. into page ($+\hat{\mathbf{y}}$)

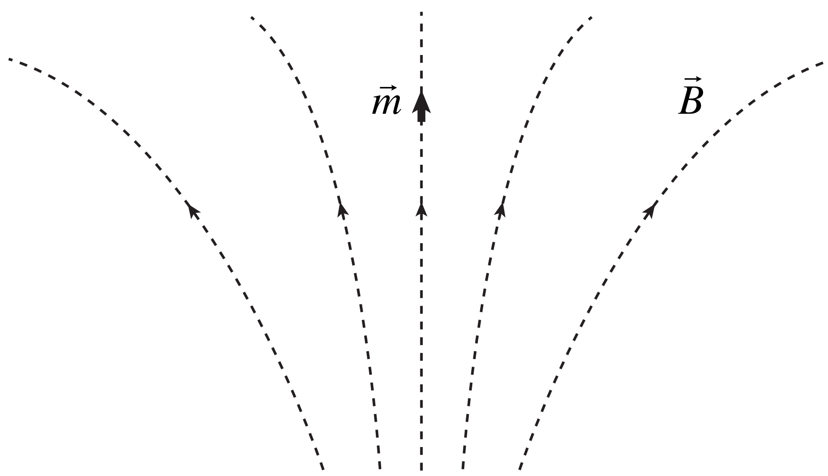
D. A small loop of wire in the xy plane carries a current that gives it a dipole moment $\mathbf{m} = m\hat{\mathbf{z}}$, with $m > 0$.The loop rests in a magnetic field as shown below, which decreases in magnitude as z increases. The direction of the force on the loop is

a. no direction, force is 0

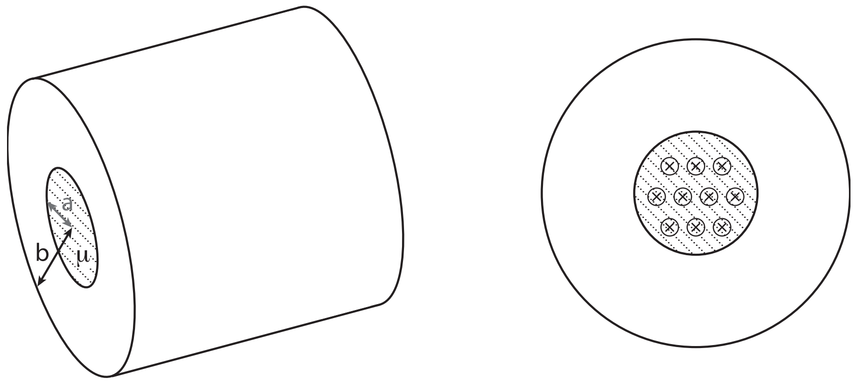
b. $+\hat{\mathbf{z}}$

c. $\boxed{-\hat{\mathbf{z}}}$

d. insufficient information to determine



2. **28 points**.
A coaxial cable has a solid inner cylindrical conductor, radius a, carrying a free current I and an outer cylindrical shell, radius b, in which a free current $-I$ returns. The inner conductor is a diamagnetic material of magnetic permeability μ. The current in the inner conductor is distributed uniformly throughout. The region between the cables is air, with $\mu = \mu_0$.



A. The current in the inner conductor is directed into the page, as shown at right. Sketch **H** on that diagram indicating its direction.
[Solution: $r < a,\ H = \frac{Ir}{2\pi a^2},\ a < r < b,\ H = I/2\pi r, r > b,\ H = 0.$ in the clockwise direction]

B. Find the magnitudes of the H and B fields in the regions $r < a, a < r < b$, and $r > b$. Sketch both results, indicating all relevant boundaries and values.
[Solution: $\mathbf{B} = \mu\mathbf{H}, \quad (r < a).\mathbf{B} = \mu_0\mathbf{H}, \quad (r > a).$]

C. Find all of the bound currents in this problem, including the directions in which they flow. (*Hint: what should the total bound current be?*)
[Solution:

$$\mathbf{M} = \mathbf{H} - \frac{1}{\mu_0}\mathbf{B} = (1 - \frac{\mu}{\mu_0})\mathbf{H}$$

$$\mathbf{J}_b = \nabla \times \mathbf{M} = \frac{I}{\pi a^2}(1 - \frac{\mu}{\mu_0})\hat{\mathbf{z}}$$

$$\mathbf{K_b} = \mathbf{M} \times \hat{\mathbf{n}} = -\frac{I}{2\pi a}(1 - \frac{\mu}{\mu_0})\hat{\mathbf{z}}$$

The total bound current is

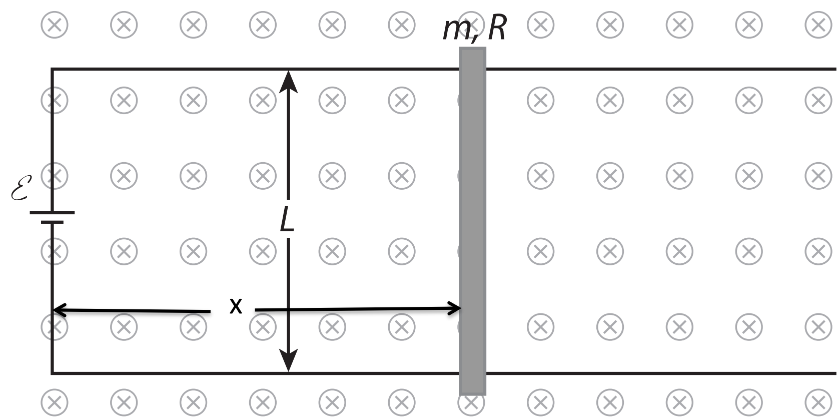
$$\mathbf{J}\pi a^2 + \mathbf{K}(2\pi a) = 0.]$$

3. [**28 points**]

You are investigating options for a rail launcher for the Hawaii Space Flight Laboratory. Your first simple model is illustrated below a conducting bar of mass m and resistance R that slides without friction on two parallel conducting rails separated by a distance L. The bar is initially at rest. The rails sit in a uniform magnetic field **B** directed into the page. A battery with voltage $\mathcal{E}$ is attached across the rails at time $t = 0$. From the battery alone, the current would circulate clockwise in the resulting circuit.

A. Suppose at a time $t > 0$, the bar moves with velocity v to the right. Find the total current flowing through the bar.
[Solution:

$$I = V/R = (\mathcal{E} - vLB)/R]$$



B. Find the equation of motion for the bar, and express it in terms of the bars velocity and the derivative dv/dt.
[Solution:

$$F = \frac{(\mathcal{E} - vBL)}{R} BL = m\frac{dv}{dt}.]$$

C. Show that the velocity will vary like $v(t) = v_{\text{final}}(1 - e^{-t/\tau})$, and find the asymptotic velocity v_{final} and the time constant τ.
[Solution:

$$\int \frac{dv}{\mathcal{E} - vLB} = \int \frac{BL}{Rm} dt$$

$$\ln\left(\frac{\mathcal{E} - vLB}{\mathcal{E}}\right) = -\frac{B^2L^2}{Rm} t$$

$$\mathcal{E} - vLB = \mathcal{E}e^{-t/\tau}, \quad \tau = \frac{Rm}{B^2L^2}$$

$$v = \frac{\mathcal{E}}{LB}(1 - e^{-t/\tau})$$

Final velocity when $\mathcal{E} = vLB$.]

D. **Bonus 10 pts**: For your second attempt, you replace the battery with a conducting wire, and change the magnitude of the applied magnetic field so that $B(t) = B_0 e^{\alpha t}$.

i. Should α be positive or negative? (Circle correct)
[Solution: We want the induced current to flow clockwise ($\mathcal{E} > 0$) so that the force on the bar is to the right.

$$\mathcal{E}(t = 0) = -\frac{\partial \Phi}{\partial t}(t = 0) = -\frac{\partial AB}{\partial t}(t = 0) = -B_0 L (\alpha x + \dot{x})$$

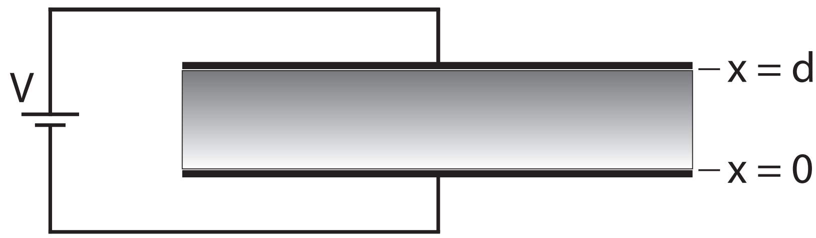
α must be negative.]

ii. Repeat part A to find the current in the bar at a time $t > 0$.
[Solution:

$$\mathcal{E} = -\frac{\partial \Phi}{\partial t} = -\frac{\partial AB}{\partial t} = B_0 e^{\alpha t} L (\alpha x + \dot{x})$$

$$I = \frac{\mathcal{E}}{R} = \frac{B_0 e^{\alpha t} L}{R} (\alpha x + \dot{x})]$$

4. [**28 points**] A parallel plate capacitor is filled with a dielectric whose dielectric constant ϵ_r varies linearly from $\epsilon_r = 1$ at the bottom plate ($x = 0$) to $\epsilon_r = 2$ at the top plate ($x = d$). A battery maintains a voltage V across the plates. Ignore edge effects in the following (ie., treat the plates as infinite in size).



A. Provide a brief justification that the displacement field $\mathbf{D}$ is uniform between the plates.
[Solution:

$$\int \mathbf{D} \cdot d\mathbf{a} = \rho_f, \quad D = \sigma_f]$$

B. Show that the displacement field between the plates is related to the potential difference across the plates by

$$D = \frac{\epsilon_0 V}{d \ln 2}$$

[Solution:

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{\mathbf{D}}{\epsilon_0 \epsilon_r(x)} = \frac{\mathbf{D}}{\epsilon_0(1 + x/d)}$$

The voltage between the plates is

$$V = \int_0^d \mathbf{E} \cdot d\mathbf{l} = \int_0^d \frac{D}{\epsilon_0(1 + x/d)} dx$$

$$V = \frac{dD}{\epsilon_0} \ln(1 + x/d) \mid_0^d = \frac{dD}{\epsilon_0} \ln 2$$

$$D = \frac{e_0 V}{d \ln 2}]$$

C. Find all of the bound surface and volume charges densities.
[Solution: The total charge density (the free charge is only on the surface)

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E} = \epsilon_0 \nabla \cdot \frac{\mathbf{D}}{\epsilon_0(1 + x/d)}$$

$$\rho_b = -\frac{D}{d(1 + x/d)^2}$$

The total bound charge

$$Q_b = \int_0^d \rho_b A dx = A \frac{D}{(1 + x/d)} \mid_0^d = A \frac{D}{2} = A \frac{\sigma_f}{2}$$

The bound surface charge is

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = (\mathbf{D} - \epsilon_0 \mathbf{E}) \cdot \hat{\mathbf{n}}$$

On the surface at $x = 0$, $\sigma_b = 0$. On the surface at $x = d$, $\sigma_b = \sigma_f - \frac{1}{2}\sigma_f = -\frac{1}{2}\sigma_f$]

D. What is the total bound charge ? (*Show your work*)
[Solution: The total bound charge is $\sigma_{tot} = A\sigma_b(x = 0) + A\sigma_b(x = d) + Q_b = 0.$]