PHYS 3323 - Formulae

$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\, d\tau'$ | $\oint \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\epsilon_0}Q_{enc}$ | $\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$

$\oint \mathbf{E}\cdot d\mathbf{l} = 0$ | $\nabla\times\mathbf{E} = 0$ | $\mathbf{E} = -\nabla V$

$V(\mathbf{r}) = -\int_O^r \mathbf{E}(\mathbf{r}')\cdot d\mathbf{l}$ | $\nabla^2 V = -\frac{\rho}{\epsilon_0}$ | $V(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{\mathscr{r}}d\tau'$

$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0}$

At the surface of a conductor $\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$ and $\sigma = -\epsilon_0\frac{\partial V}{\partial n}$

$C = Q/V$ | $W = \frac{1}{2}CV^2$ | $\mathbf{F} = q\mathbf{E}$

$V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}\cdot\int \mathbf{r}'\rho(\mathbf{r}')d\tau$ | $\mathbf{p} = \int \mathbf{r}'\rho(\mathbf{r}')d\tau'$ | $V_{dip}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$

$\mathbf{E}_{dip} = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta})$

Solution to Laplace equation in cartesian coordinates in two dimensions

$$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky).$$

Solution to Laplace equation in spherical coordinates

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta).$$

Solution to Laplace equation in cylindrical coordinates (2 D)

$$V(\rho,\phi) = \sum_{m=1}^{\infty}(A_m\cos m\phi + B_m\sin m\phi)(C_m\rho^m + \frac{D_m}{\rho^m}) + A_0 + B_0\ln(\rho)$$

Multipole expansion

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')d\tau'$$

- Spherical:
  - line element
$$d\mathbf{l} = dr\hat{\mathbf{r}} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$$
  - Gradient:
$$\nabla\mathbf{t} = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial\phi}\hat{\phi}$$
  - Divergence:
$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$
- Cylindricall:
  - Gradient:
$$\nabla\mathbf{t} = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial\phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$
  - Divergence:
$$\nabla\cdot\mathbf{v} = \frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$
  - Curl:
$$\nabla\times\mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\phi} + \frac{1}{r}\left(\frac{\partial(rv_\phi)}{\partial r} - \frac{\partial v_r}{\partial\phi}\right)\hat{\mathbf{z}}$$

Electric Displacement

$$\rho = \rho_b + \rho_f.$$

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho = -\nabla \cdot \mathbf{P} + \rho_f.$$

$$\rho_f = \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}).$$

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}.$$

$$\nabla \cdot \mathbf{D} = \rho_f.$$

$$\nabla \cdot \mathbf{P} = -\rho_b$$

$$\mathbf{P} \cdot \hat{\mathbf{n}} = \sigma_b$$

Linear dielectric

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E}$$

where χ is the electric suscpetiblity.

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 (1 + \chi) \mathbf{P} = \epsilon \mathbf{E}.$$

Boundary conditions

$$\begin{aligned} D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} &= \sigma_f \qquad \mathbf{D}^{\parallel}_{\text{above}} - \mathbf{D}^{\parallel}_{\text{below}} = \mathbf{P}^{\parallel}_{\text{above}} - \mathbf{P}^{\parallel}_{\text{below}} \\ E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} &= \frac{1}{\epsilon_0}\sigma \qquad \mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = 0 \end{aligned}$$

In linear dielectic

$$\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot \left(\epsilon_0 \frac{\chi}{\epsilon} \mathbf{D} \right) = -\left(\frac{\chi}{1 + \chi} \right) \rho_f.$$

the bound charge density is proportional to the free charge density.

$$\epsilon_{above} E^{\perp}_{above} - \epsilon_{below} E^{\perp}_{below} = \sigma_f.$$

Current

$$\mathbf{J} = \sigma \mathbf{E} \qquad V = IR$$

$$\mathbf{F} = \int (\mathbf{v} \times \mathbf{B}) dq = \int (\mathbf{v} \times \mathbf{B}) \lambda dl = \int (\mathbf{I} \times \mathbf{B}) dl.$$

$$\mathbf{F}_{mag} = I \int (d\mathbf{l} \times \mathbf{B}).$$

$$F_{mag} = \int (\mathbf{v} \times \mathbf{B}) \rho d\tau = \int (\mathbf{J} \times \mathbf{B}) d\tau.$$

$$\nabla \cdot \mathbf{J} \equiv -\frac{\partial \rho}{\partial t}.$$

Magnetic Field

Biot-Savart

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dl' = \frac{\mu_0}{4\pi}I\int\frac{d\mathbf{l}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}.$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}d\tau'.$$

Divergence

$$\nabla\cdot\mathbf{B} = \frac{\mu_0}{4\pi}\int\nabla\cdot\left(\frac{\mathbf{J}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)d\tau'.$$

$$\nabla\cdot\mathbf{B} = 0.$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$$\oint\mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{enc} + \mu_0\epsilon_0\frac{\partial\Phi_e}{\partial t}.$$

Vector potential

$$\mathbf{B} = \nabla\times\mathbf{A}.$$

$$\nabla\times\mathbf{B} = \nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} = \mu_0\mathbf{J}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}d\tau'.$$

$$\mathbf{A} = \frac{\mu_0 I}{4\pi}\int\frac{1}{r}d\mathbf{l}'.$$

Boundary Conditions

In linear media

$$\begin{aligned}\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp &= \sigma_f, \qquad \mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0\\ B_1^\perp - B_2^\perp &= 0, \qquad \frac{1}{\mu_1}\mathbf{B}_1^\parallel - \frac{1}{\mu_2}\mathbf{B}_2^\parallel = \mathbf{K_f}\times\hat{\mathbf{n}}\end{aligned}$$

If there is no free charge or free current at the interface

$$\begin{aligned}\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp &= 0, \qquad \mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0\\ B_1^\perp - B_2^\perp &= 0, \qquad \frac{1}{\mu_1}\mathbf{B}_1^\parallel - \frac{1}{\mu_2}\mathbf{B}_2^\parallel = 0\end{aligned}$$

The vector potential is continuous at a boundary.

Multipole Expansion

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int \frac{1}{\mathscr{r}} d\mathbf{l}' = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\alpha) d\mathbf{l}'.$$

$$\mathbf{m} = I\int d\mathbf{a}'$$

$$\mathbf{A}_{\mathbf{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}.$$

$$\mathbf{A}_{\mathbf{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}.$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}).$$

Torque and Force

$$\mathbf{N} = \mathbf{m}\times\mathbf{B}.$$

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$$

Magnetization

The total current

$$\mathbf{J} = \mathbf{J_b} + \mathbf{J_f}$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} = \mu_0(\nabla\times\mathbf{M} + \mathbf{J_f})$$

$$\nabla\times(\frac{\mathbf{1}}{\mu_\mathbf{0}}\mathbf{B} - \mathbf{M}) = \mathbf{J_f}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

$$\nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t}$$

$$\oint \mathbf{H}\cdot d\mathbf{l} = I_{enc}^{free} + \frac{d}{dt}\int_S \mathbf{D}\cdot d\mathbf{a}$$

$$\mathbf{J}_b = \nabla\times\mathbf{M} \qquad\qquad \mathbf{K_b} = \mathbf{M}\times\hat{\mathbf{n}}$$

Linear media:

$$\mathbf{M} = \chi_m\mathbf{H}$$

$$\chi_m = \frac{\mu}{\mu_0} - 1$$

$$\mathbf{B} = \mu\mathbf{H}$$

Boundary conditions:

$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}} \qquad B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K$$

$$\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}} \qquad \mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K}\times\hat{\mathbf{n}})$$

Induced fields

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int \mathbf{B} \cdot \mathbf{da}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi} \int \frac{-\frac{\partial \mathbf{B}}{\partial t} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^{2}} d\tau' = -\frac{1}{4\pi} \frac{\partial}{\partial t} \int \frac{\mathbf{B} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^{2}} d\tau$$

$$\int \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{l} = \mu_0 I_{enc}$$

Inductance

$$\Phi_2 = M_{21} I_1 \qquad M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\mathscr{r}}$$

$$M_{21} = M_{12}, \qquad \mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M \frac{dI_1}{dt}$$

$$\Phi = LI, \qquad \mathcal{E} = -L \frac{dI}{dt}$$

Maxwell Equations

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{1}{\epsilon_0} \rho & \qquad \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

In matter

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_f & \qquad \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & \qquad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{aligned}$$

Wave equation

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \qquad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

$$v = \frac{1}{\sqrt{\epsilon \mu}}$$

Waves and Waveguides

Phase velocity and group velocity (k is the propagation direction)

$$v_p = \frac{\omega}{k}, \qquad v_g = \frac{d\omega}{dk}$$

Rectangular waveguide

$$k = \sqrt{(\omega/c)^2 - \pi^2[(m/a)^2 + (n/b)^2]}, \qquad \omega_{mn} = c\pi\sqrt{(m/a)^2 + (n/b)^2}, \qquad k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}$$

Energy and Momentum of EM fields

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right), \qquad \mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$$

Momentum

$$\mathbf{g} = \epsilon_0(\mathbf{E}\times\mathbf{B}), \qquad \mathbf{p} = \int_V \mathbf{g}\,d\tau$$

Angular momentum

$$\mathbf{l} = \mathbf{r}\times\mathbf{g} = \epsilon_0[\mathbf{r}\times(\mathbf{E}\times\mathbf{B})]$$

Legendre Polynomials

$$\begin{aligned}
P_0(\cos\theta) &= 1 \\
P_1(\cos\theta) &= \cos\theta \\
P_2(\cos\theta) &= (3\cos^2\theta - 1)/2 \\
P_3(\cos\theta) &= (5\cos^3\theta - 3\cos\theta)/2 \\
P_4(\cos\theta) &= (35\cos^4\theta - 30\cos^2\theta + 3)/8 \\
P_5(\cos\theta) &= (63\cos^5\theta - 70\cos^3\theta + 15\cos\theta)/8
\end{aligned}$$

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \int_0^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta d\theta = \frac{2}{2l+1}\delta_{ll'}$$

$$\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)dy = \frac{1}{2}\delta_{nn'}$$