Physics 3323, Fall 2016 Problem Set 11 *due Nov 4, 2014*

Reading: Griffiths Ch. 7 & 8

1. Time-varying solenoid

Griffiths 7.12. A long solenoid, of radius a, is driven by an alternating current, so that the field inside the solenoid: $\mathbf{B}(t) = B_0 \cos(\omega t)\hat{\mathbf{z}}$. A circular loop of wire, of radius $a/2$ and resistance R, is placed inside the solenoid, and coaxial with it. Find the current induced in the loop, as a function of time.

SOLUTION: The flux through the loop is

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} = B_0 \cos(\omega t)\pi \frac{a^2}{4}$$

$$\mathcal{E} = -\frac{d\Phi}{dt} = B_0 \frac{\pi a^2}{4} \omega \sin(\omega t) = IR.$$

$$\rightarrow I = B_0 \frac{\pi a^2}{4R} \omega \sin(\omega t).$$

2. Square loops

Griffiths 7.23 (formerly 7.21). A square loop of wire, of side a lies midway between two long wires, $3a$ apart, and in the same plane. (Actually, the long wires are sides of a large rectangular loop, but the short ends are so far away that they can be neglected.) A clockwise current I in the square loop is gradually increasing: $dI/dt = k$ (a constant). Find the emf induced in the big loop. Which way will the induced current flow?

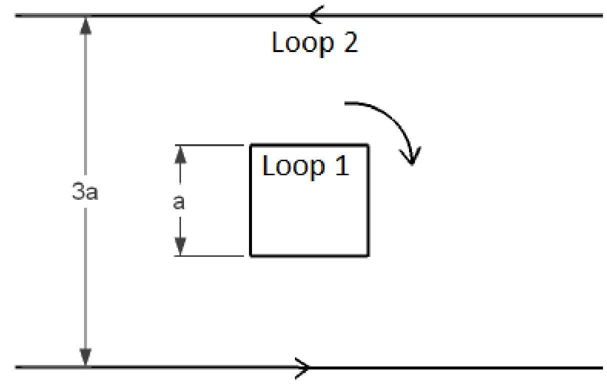


Figure 1: Square Loop in the Big Loop

SOLUTION: The EMF induced in the big loop will have magnitude $\mathcal{E} = \frac{d\Phi}{dt}$. Its direction is given by Lenz's law. Since the clockwise current in the small loop is increasing, the flux into the page is increasing. This means that the induced current in the big loop will create flux out of the page and will be counter-clockwise. Lets label the big loop '2' and the small loop '1'.

$$\mathcal{E}_2 = \frac{d\Phi_2}{dt} = M_{21} \frac{dI_1}{dt} \tag{2.1}$$

Now M_{21} is hard to calculate, since the magnetic field from the small loop is hard to calculate. But the flux through the small loop from the large loop is easier, since the large loop is just two long straight currents. The contribution to the flux from each current is the same. We can use the relation $M_{21} = M = M_{12}$ to calculate the mutual inductance.

$$M = \frac{\Phi_1}{I_2} = 2\int_a^{2a} \frac{\mu_0}{2\pi s} a\, ds = \frac{\mu_0 a}{\pi} \ln 2 \tag{2.2}$$

Now $\frac{dI_1}{dt} = k$, so

$$\boxed{\mathcal{E}_2 = \frac{\mu_0 a k \ln 2}{\pi}} \tag{2.3}$$

3. Conductive capacitor

A thin metal plate capacitor of plate separation d is filled with a medium of conductivity σ and dielectric constant ε. The plates of the capacitor are circular. A variable voltage $V = V_0 \sin \omega t$ is applied to the capacitor a shown in the figure. Assuming that the electric field between the plates is homogeneous, find H in the capacitor.

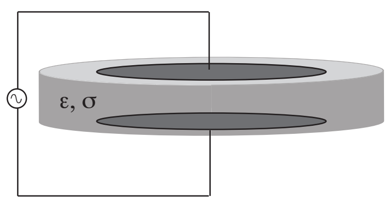


Figure 2: Conductive Capacitor Arrangement

SOLUTION: Since the electric field is homogeneous, we can simply write it down as follows.

$$\vec{E} = \frac{V}{d}\,\hat{\mathbf{z}} = \frac{V_0}{d} \sin \omega t\, \hat{\mathbf{z}} \tag{3.1}$$

Now we can use Ampere's law

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \tag{3.2}$$

along with the constitutive relation $\vec{D} = \varepsilon \vec{E}$ and ohms law $\vec{J} = \sigma \vec{E}$ to get $\vec{H}$. The cylindrical symmetry requires $\vec{H} = H\hat{\phi}$. Consider an Amperian loop centered on the axis of symmetry with radius s.

$$\oint \vec{H} \cdot d\vec{\ell} = \int \left[\vec{J}_f + \frac{\partial \vec{D}}{\partial t}\right] \cdot d\vec{a} = \int \left[\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right] \cdot d\vec{a} \tag{3.3}$$

$$H(2\pi s) = \frac{V_0}{d} \left(\sigma \sin \omega t + \varepsilon\omega \cos \omega t\right) \pi s^2 \tag{3.4}$$

$$\boxed{\vec{H} = \frac{V_0 s}{2d} \left(\sigma \sin \omega t + \varepsilon\omega \cos \omega t\right)\, \hat{\phi}} \tag{3.5}$$

We see that the answer has a resistive component in phase with the voltage, and a capacitive component out of phase with the voltage by $\pi/2$.

4. Cable Inductance

Consider a coaxial cable consisting of two long concentric hollow conducting cylinders with radii a and b. A current I travels up the inner cylinder, and returns down the outer cylinder. Determine the self-inductance per unit length, both from using our identity $L = \Phi/I$, and from comparing the magnetic energy with the standard circuit form $\frac{1}{2}LI^2$.

SOLUTION: The magnetic field is circumferential, so the relevant flux is that which passes between radius a and b. The magnetic field inside the cable is $\vec{B} = \frac{\mu_0 I}{2\pi s}$. First we'll find the inductance per unit length by using the identity $L = \Phi/I$.

$$\Phi = \int \vec{B} \cdot d\vec{a} = \frac{\mu_0 I}{2\pi} \int_a^b \frac{\ell\, ds}{s} = \frac{\mu_0 I \ell}{2\pi} \ln \frac{b}{a} \tag{4.1}$$

$$\boxed{\frac{L}{\ell} = \frac{\Phi}{I\ell} = \frac{\mu_0}{2\pi} \ln \frac{b}{a}} \tag{4.2}$$

Now we will derive the same formula by comparing the magnetic energy with the standard circuit form $U = \frac{1}{2}LI^2$.

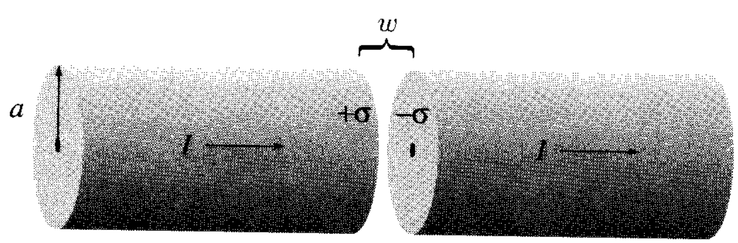
$$U = \frac{1}{2\mu_0} \int B^2\, dV = \frac{1}{2\mu_0} \int_a^b \frac{\mu_0^2 I^2}{4\pi^2 s^2} 2\pi s \ell\, ds \tag{4.3}$$

$$U = \frac{\mu_0 I^2 \ell}{4\pi} \int_a^b \frac{ds}{s} = \frac{\mu_0 I^2 \ell}{4\pi} \ln \frac{b}{a} \tag{4.4}$$

$$\boxed{\frac{L}{\ell} = \frac{2U}{I^2 \ell} = \frac{\mu_0}{2\pi} \ln \frac{b}{a}} \tag{4.5}$$

5. Induced magnetic field

A fat wire, radius a, carries a constant current I, uniformly distributed over its cross section. A narrow gap in the wire, of width $w \ll a$, forms a parallel-plate capacitor. Find the magnetic field in the gap, at a distance $s < a$ from the axis.



SOLUTION: The charge on the surface $\sigma(t) = \sigma_0 + \frac{I}{\pi a^2}$ and the electric field in the gap

$$\mathbf{E} = \frac{\sigma(t)}{\epsilon_0} \tag{5.1}$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \int \mathbf{E} \cdot d\mathbf{a} \tag{5.2}$$

$$\rightarrow \mathbf{B}(s) = \mu_0 I \frac{s}{2\pi a^2} \hat{\phi} \tag{5.3}$$

6. From standing waves to travelling

a) Write the traveling wave $\Psi(z,t) = A\cos(\omega t - kz)$ as a superposition of two standing waves.

b) Write the standing wave $\Psi(z,t) = A\cos\omega t \cos kz$ as a superposition of two traveling waves that travel in opposite directions.

c) Rewrite the following superposition of two traveling waves

$$\Psi(z,t) = A\cos(\omega t - kz) + RA\cos(\omega t + kz),$$

where R is also a constant, as a superposition of standing waves.

SOLUTION: These problems are applications of the following identities:

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta \tag{6.1}$$

$$\cos\alpha \cos\beta = \frac{1}{2}\left[\cos(\alpha + \beta) + \cos(\alpha - \beta)\right] \tag{6.2}$$

a)

$$\Psi(z,t) = A\cos(\omega t - kz) \tag{6.3}$$

$$= A\cos\omega t\cos kz + A\sin\omega t\sin kz \tag{6.4}$$

b)

$$\Psi(z,t) = A\cos\omega t\cos kz \tag{6.5}$$

$$= \frac{A}{2}\left[\cos(\omega t + kz) + \cos(\omega t - kz)\right] \tag{6.6}$$

c)

$$\Psi(z,t) = A\cos(\omega t - kz) + RA\cos(\omega t + kz) \tag{6.7}$$

$$= A[\cos\omega t\cos kz + \sin\omega t\sin kz] + RA[\cos\omega t\cos kz - \sin\omega t\sin kz] \tag{6.8}$$

$$= (1+R)A\cos\omega t\cos kz + (1-R)A\sin\omega t\sin kz \tag{6.9}$$

7. Am I a wave?

Griffiths 9.1. By explicit differentiation, check that the functions

$$f_1(z,t) = Ae^{-b(z-vt)^2}, \quad f_2(z,t) = A\sin[b(z-vt)],$$

$$\text{and} \quad f_3(z,t) = \frac{A}{b(z-vt)^2+1}$$

satisfy the wave equation and

$$f_4(z,t) = Ae^{-b(bz^2+vt)}, \quad \text{and} \quad f_5(z,t) = A\sin(bz)\cos(bvt)^3$$

do not.

SOLUTION:

1.

$$\frac{\partial^2 f_1}{\partial z^2} = \left[4b^2(z-vt)^2 - 2b\right]Ae^{b(z-vt)^2} \tag{7.1}$$

$$\frac{\partial^2 f_1}{\partial t^2} = \left[4b^2(z-vt)^2 - 2b\right]v^2Ae^{b(z-vt)^2} = v^2\frac{\partial^2 f_1}{\partial z^2} \tag{7.2}$$

2.

$$\frac{\partial^2 f_2}{\partial z^2} = -b^2A\sin[b(z-vt)] \tag{7.3}$$

$$\frac{\partial^2 f_2}{\partial t^2} = -(bv)^2A\sin[b(z-vt)] = v^2\frac{\partial^2 f_2}{\partial z^2} \tag{7.4}$$

3.

$$\frac{\partial^2 f_3}{\partial z^2} = \frac{2Ab}{(b(z-vt)^2+1)^2}\left[4b(z-vt)^2 - 1\right] \tag{7.5}$$

$$\frac{\partial^2 f_3}{\partial t^2} = \frac{2Abv^2}{(b(z-vt)^2+1)^2}\left[4b(z-vt)^2 - 1\right] = v^2\frac{\partial^2 f_3}{\partial t^2} \tag{7.6}$$

4.

$$\frac{\partial^2 f_4}{\partial z^2} = 2b^2\left[2b^2z^2 - 1\right]Ae^{-b(bz^2+vt)} \tag{7.7}$$

$$\frac{\partial^2 f_4}{\partial t^2} = b^2v^2Ae^{-b(bz^2+vt)} \neq v^2\frac{\partial^2 f_4}{\partial z^2} \tag{7.8}$$

5.

$$\frac{\partial^2 f_5}{\partial z^2} = -b^2A\sin(bz)\cos^3(bvt) \tag{7.9}$$

$$\frac{\partial^2 f_5}{\partial t^2} = 3A(bv)^2\left[2\cos(bvt)\sin^2(bvt) - cos^3(bvt)\right] \neq v^2\frac{\partial^2 f_5}{\partial z^2} \tag{7.10}$$

8. Circular polarization on a string

Griffiths 9.8. Equation 9.36 describes the most general **linearly** polarized wave on a string . Linear (or "plane") polarization (so called because the displacement is parallel to a fixed vector $\hat{\mathbf{n}}$) results from the combination of horizontally and vertically polarized waves of the *same phase* (Eq. 9.39). If the two components are of equal amplitude, but *out of phase* by 90° (say, $\delta_v = 0, \delta_h = 90°$), the result is a *circularly* polarized wave. In that case:

(a) At a fixed point z, show that the string moves in a circle about the z axis. Does it go *clockwise or counterclockwise*, as you look down the axis toward the origin? How would you construct a wave circling the *other* way? (In optics, the clockwise case is called **right circular polarization**, and the conterclockwise, **left circular polarization**.)

(b) Sketch the string at time $t = 0$.

(c) How would you shake the string in order to produce a circularly polarized wave?

SOLUTION: We can write this wave as a superposition of two perpendicular linearly polarized waves with the same amplitude, but out of phase by $\pi/2$.

$$\tilde{\mathbf{f}}(z,t) = \left(\tilde{A}_x\,\hat{x} + \tilde{A}_y\,\hat{y}\right) e^{i(kz-\omega t)} \tag{8.1}$$

$$= A\left(\hat{x} + e^{i\pi/2}\,\hat{y}\right) e^{i(kz-\omega t)} \tag{8.2}$$

$$\tilde{\mathbf{f}}(z,t) = A\,(\hat{x} + i\,\hat{y})\, e^{i(kz-\omega t)} \tag{8.3}$$

a) Let's pick the point $z = 0$ to see which way the vector rotates.

$$\tilde{\mathbf{f}}(z,t) = A\,(\hat{x} + i\,\hat{y})\, e^{-i\omega t} = A\,(\hat{x} + i\,\hat{y})\,(\cos(\omega t) - i\sin(\omega t)) \tag{8.4}$$

Remember that the actual vector is the real part of Eq. (8.4). This gives the following parametrization.

$$f_x = A\cos(\omega t) \tag{8.5}$$

$$f_y = A\sin(\omega t) \tag{8.6}$$

This is a parametric equation for a circle of radius A rotating counter-clockwise as t increases. To get a wave that circles the other way, we could start with $\tilde{A}_y = Ae^{-i\pi/2}$ instead of the positive phase difference.

b) See Figure 3

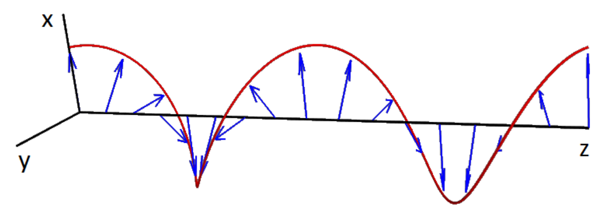


Figure 3: Plot of circular wave at $t = 0$

c) To create this wave, you could just grab one end of the string and swing your arm in a circle.

9. Hey, don't push me around!

Griffiths 9.10. The intensity of sunlight hitting the earth is about 1300 W/m^2. If sunlight strikes a perfect absorber what pressure does it exert? How about a perfect reflector? What fraction of atmospheric pressure does this amount to?

SOLUTION: For a perfect absorber $P = I/c$. With $I = 1300\ \mathrm{W/m^2}$ we have:

$$\boxed{P = I/c = 4.34 \times 10^{-6}\,\mathrm{Pa}} \tag{9.1}$$

A perfect reflector has twice the pressure

$$\boxed{P = 8.67 \times 10^{-6}\,\mathrm{Pa}} \tag{9.2}$$

These correspond to 4.3×10^{-11} atm and 8.6×10^{-11} atm respectively.