Physics 3323, Fall 2016 Problem Set 11 *due Nov 18, 2016*

Reading: Griffiths Ch. 8

1. Force and momentum (Griffiths 8.6)

A charged parallel-plate capacitor (with uniform electric field $\mathbf{E} = E\hat{\mathbf{z}}$) is placed in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{x}}$ as shown in Figure 1.

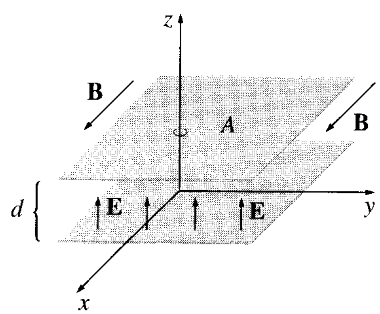

Figure 1: Parallel plate capacitor

a) Find the electromagnetic momentum in the space between the plates.

b) Now a resistive wire is connected between the plates, along the z-axis, so that the capacitor slowly disharges. The current through the wire will experience a magnetic force; what is the total impluse delivered to the system, during the discharge?

c) Instead of turning off the *electric* field (as in (b)), suppose we slowly reduce the *magnetic* field. This will induce a Farady electric field, which in turn exerts a force on the plates. Show that the total impulse is (again) equal to the momentum originally stored in the fields.

SOLUTION:

a) The electromagnetic momentum

$$\mathbf{p} = \epsilon_0 \int \mathbf{E} \times \mathbf{B} d\tau = \epsilon_0 EBAd\hat{\mathbf{y}}$$

b) The force on the wire is

$$\mathbf{F} = IdB\hat{\mathbf{y}} = \frac{dQ}{dt} dB\hat{\mathbf{y}}$$

$$\mathbf{p} = \mathbf{F}dt = QdB\hat{\mathbf{y}}$$

The electric field $E = \frac{Q}{A\epsilon_0}$ Then

$$\mathbf{p} = \epsilon_0 EAdB\hat{\mathbf{y}}$$

c) The magnetic flux $\Phi = \int \mathbf{B} \cdot d\mathbf{a} = Bdw$ where w is the width of the plate. The change in flux through the loop across the width of the top plate, the distance d separating the plates, backwards across the width of the bottom plate and across the gap to the starting point corresponds to

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = 2Ew = \frac{\partial \Phi}{\partial t} = dw\frac{\partial B}{\partial t}$$

The force on the plates is

$$F = E(Q_{top} + Q_{bot}) = dQ\frac{\partial B}{\partial t} = dA\epsilon_0 E\frac{\Delta B}{\Delta t}\hat{\mathbf{y}}$$

$$\mathbf{p} = \mathbf{F}\Delta t = dA\epsilon_0 EB\hat{\mathbf{y}}$$

2. Electron (Griffiths 8.11)

Picture the electron as a uniformly charged spherical shell, with charge e and radius R, spinning at angular velociy ω. (See Example 5.11 for the vector potential of the spinning shell)

a) Calculate the total energy contained in the electromagnetic fields.

b) Calculate the total angular momentum contained in the fields.

c) According to the Einstein formula ($E = mc^2$), the energy in the fields should contribute to the mass of the electron. Lorentz and others speculated that the *entire* mass of the electron might be accounted for in this way: $U_{em} = m_e c^2$. Suppose, moreover that the electron's spin angular momentum is entirely attributable to the electromagnetic fields: $L_{em} = \hbar/2$. On these two assumptions, determine the radius and angular velocity of the electron. What is their product, ωR? Does this classical model make sense?

SOLUTION:

a) The electric field outside the sphere is

$$\mathbf{E} = \frac{e}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}$$

The vector potential outside is

$$\mathbf{A} = \frac{\mu_0 R^4 \omega\sigma}{3}\frac{\sin\theta}{r^2}\hat{\phi}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{\mu_0 R^4\omega\sigma}{3}\left(\frac{2\cos\theta}{r^3}\hat{\mathbf{r}} + \frac{\sin\theta}{r^3}\hat{\theta}\right)$$

And inside

$$\mathbf{B} = \frac{2}{3}\mu_0\sigma R\omega\hat{\mathbf{z}}.$$

Then the total energy outside the sphere is

$$\begin{aligned}
U_{out} &= \frac{1}{2}\int\left(\epsilon_0\frac{e^2}{16\pi^2\epsilon_0^2 r^4}r^2 dr 2\pi\sin\theta d\theta\right) + \\
&\quad \frac{1}{\mu_0}\left(\frac{\mu_0 R^4\omega\sigma}{3}\right)^2\left(\frac{4\cos^2\theta}{r^6} + \frac{\sin^2\theta}{r^6}\right)2\pi r^2 dr\sin\theta d\theta \\
&= \frac{1}{2}\left(\frac{e^2}{4\pi\epsilon_0 R} + \frac{2\pi}{\mu_0}\left(\frac{\mu_0 R^4\omega\sigma}{3}\right)^2\left(\frac{4}{3R^3}\right)\right) \\
&= \frac{1}{2}\left(\frac{e^2}{4\pi\epsilon_0 R} + \frac{2\pi}{\mu_0}\left(\frac{\mu_0 e\omega}{3(4\pi)}\right)^2\left(\frac{4R}{3}\right)\right) \\
&= \frac{1}{2}\left(\frac{e^2}{4\pi\epsilon_0 R} + \mu_0\frac{e^2\omega^2}{2\pi}\frac{R}{27}\right) \\
&= \frac{e^2}{4\pi\epsilon_0 R}\left(\frac{1}{2} + \frac{\omega^2 R^2}{27c^2}\right)
\end{aligned}$$

And the energy inside the sphere is

$$U_{in} = \frac{1}{2\mu_0}\frac{4}{9}(\mu_0\sigma R\omega)^2\frac{4\pi}{3}R^3 = \frac{\mu_0}{4\pi}\frac{2}{27}e^2\omega^2 R = \frac{e^2}{4\pi\epsilon_0 R}\frac{\omega^2 R^2}{27c^2}$$

b) The angular momentum is

$$\begin{aligned}
\mathbf{L} &= \epsilon_0 \int (\mathbf{r}\times\mathbf{E}\times\mathbf{B}) r^2 dr \sin\theta d\theta d\phi \\
&= 2\pi \int \left(\frac{e}{4\pi}\right)^2 \frac{\mu_0 R^2\omega}{3} \frac{\sin\theta}{r^4} (\hat{\mathbf{r}}\times\hat{\phi}) r^2 dr \sin\theta d\theta \\
&= 2\pi \int \left(\frac{e}{4\pi}\right)^2 \frac{\mu_0 R^2\omega}{3} \frac{\sin\theta}{r^4} \sin\theta \hat{\mathbf{z}} r^2 dr \sin\theta d\theta \\
&= 2\pi \int \left(\frac{e}{4\pi}\right)^2 \frac{\mu_0 R^2\omega}{3} \frac{4}{3r^4} \hat{\mathbf{z}} r^2 dr \\
&= \frac{e^2}{2\pi} \frac{\mu_0\omega}{9} R\hat{\mathbf{z}}
\end{aligned}$$

3. Electromagnetic momentum again

Imagine two parallel infinite sheets, carrying uniform surface charge $+\sigma$ (on the sheet at $z = d$) and $-\sigma$ (at $z = 0$). They are moving in the y-direction at constant speed v.

a) What is the electromagnetic momentum in region of area A?

b) Now suppose the top sheet moves slowly down (speed u) until it reaches the bottom sheet, so the fields disappear. By calculating the (magnetic) force on the charge ($q = \sigma A$), show that the impulse delivered to the sheet is equal to the momentum originally stored in the fields.

SOLUTION:

a) The momentum

$$\mathbf{p} = \epsilon_0 \int \mathbf{E}\times\mathbf{B} d\tau = \epsilon_0 \frac{\sigma}{\epsilon_0} \mu_0 \sigma v (dA) \hat{\mathbf{x}}$$

b)The force on the charged plate as if moves with velocity $\frac{dz}{dt}$ through the magnetic field $\mathbf{B}$ is

$$\mathbf{F} = \sigma A \frac{dz}{dt} B$$

The change in momentum, that is the impulse, is force X time.

$$\mathbf{F} = \mathbf{p}\Delta t = \sigma A d(\mu_0 \sigma v)$$

which is indeed the momentum that was stored in the fields.

4. Toroid

A point charge q is located at the center of a toroidal coil of rectangular cross section, inner radius a, outer radius $a + w$, and height hm which carries a total of N tightly wound turns and current I.

a) Find the electromagnetic momentum $\mathbf{p}$ of this configuration, assuming that w and h are both much less than a (so you can ignore the variation of the fields over the cross section).

b) Now the current in the toroid is turned off, quickly enough that the point charge does not move appreciably as the magnetic field drops to zero. Show that the impluse imparted to q is equal to the momentum originally stored in the electromagnetic fields.

SOLUTION:

a) The electric field is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}$$

The magnetic field of the toroid is nonzero inside the volume of the toroid

$$\begin{aligned}\oint \mathbf{B}\cdot d\mathbf{l} &= \mu_0 NI \\ \rightarrow \mathbf{B} &= \frac{\mu_0 NI}{2\pi a}\hat{\phi}\end{aligned}$$

Since $a \gg w, h$, we assume the magnetic field is constant over the volume of the toroid. The electromagnetic momentum is

$$\begin{aligned}\mathbf{p} = \epsilon_0 \int \mathbf{E}\times\mathbf{B}d\tau &= \epsilon_0 \frac{q}{4\pi\epsilon_0 a^2}\frac{\mu_0 NI}{2\pi a}(2\pi ahw)\hat{\mathbf{z}} \\ &= \frac{\mu_0 qNIhw}{4\pi a^2}\hat{\mathbf{z}}\end{aligned}$$

b) To determine the induced field at q when the magnetic field in the toroid is reduced to zero we use

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

in integral form

$$\begin{aligned}\mathbf{E} &= \frac{-1}{4\pi}\frac{\partial}{\partial t}\int\frac{\mathbf{B}\times\hat{\boldsymbol{\imath}}}{\imath^{\,2}}d\tau \\ &= -\frac{1}{4\pi}\frac{\partial}{\partial t}\int\frac{Bdl}{a^2}hw \\ &= -\frac{1}{4\pi}\frac{\partial}{\partial t}\int\frac{B2\pi a}{a^2}hw \\ &= -\frac{1}{4\pi}\frac{\partial I}{\partial t}\frac{\mu_0 N}{a^2}hw \\ \rightarrow\quad & \mathbf{F}dt = q\mathbf{E}dt = \frac{1}{4\pi}\frac{\mu_0 NI}{a^2}hw\end{aligned}$$