Physics 3323, Fall 2016 Problem Set 13 *due* Friday, *Dec 2, 2014*

Reading: Finish Griffiths Ch. 9, and 10.2.1, 10.3, and 11.1.1-2

1. Reflecting on polarizations

Griffiths 9.15 (3rd ed.: 9.14). In writing (9.76)

$$\begin{aligned}\tilde{\mathbf{E}}_R(z,t) &= \tilde{E}_{0R}e^{i(-k_1z-\omega t)}\hat{\mathbf{x}}\\ \tilde{\mathbf{B}}_R(z,t) &= -\frac{1}{v_1}\tilde{E}_{0R}e^{i(-k_1z-\omega t)}\hat{\mathbf{y}}\end{aligned}$$

and (9.77)

$$\begin{aligned}\tilde{\mathbf{E}}_T(z,t) &= \tilde{E}_{0T}e^{i(k_2z-\omega t)}\hat{\mathbf{x}}\\ \tilde{\mathbf{B}}_T(z,t) &= \frac{1}{v_2}\tilde{E}_{0T}e^{i(k_2z-\omega t)}\hat{\mathbf{y}}\end{aligned}$$

I tacitly assume that the reflected and transmitted waves have the same *polarization* as the incident wave - along the x direction. Prove that this *must* be so. [*Hint*: Let the polarization vectors of the transmitted and relfected waves be

$$\hat{\mathbf{n}}_T = \cos\theta_T\hat{\mathbf{x}} + \sin\theta_T\hat{\mathbf{y}}, \quad \hat{\mathbf{n}}_R = \cos\theta_R\hat{\mathbf{x}} + \sin\theta_R\hat{\mathbf{y}}$$

and prove from the boundary conditions that $\theta_T = \theta_R = 0$.]

SOLUTION: If we allow the reflected and transmitted waves to have polarization rotated by angles θ_R and θ_T respectively, then the equations describing these waves must be changed from their form in Griffiths.

Incident:

$$\left.\begin{aligned}\vec{E}_I &= \tilde{E}_{0_I}e^{i(k_1z-\omega t)}\,\hat{\mathbf{x}}\\ \vec{B}_I &= \tfrac{1}{v_1}\tilde{E}_{0_I}e^{i(k_1z-\omega t)}\,\hat{\mathbf{y}}\end{aligned}\right\} \tag{1.1}$$

Reflected:

$$\left.\begin{aligned}\vec{E}_R &= \tilde{E}_{0_R}e^{i(-k_1z-\omega t)}(\cos\theta_R\,\hat{\mathbf{x}} + \sin\theta_R\,\hat{\mathbf{y}})\\ \vec{B}_R &= -\tfrac{1}{v_1}\tilde{E}_{0_R}e^{i(-k_1z-\omega t)}(-\sin\theta_R\,\hat{\mathbf{x}} + \cos\theta_R\,\hat{\mathbf{y}})\end{aligned}\right\} \tag{1.2}$$

Transmitted:

$$\left.\begin{aligned}\vec{E}_T &= \tilde{E}_{0_T}e^{i(k_2z-\omega t)}(\cos\theta_T\,\hat{\mathbf{x}} + \sin\theta_T\,\hat{\mathbf{y}})\\ \vec{B}_T &= \tfrac{1}{v_2}\tilde{E}_{0_T}e^{i(-k_2z-\omega t)}(-\sin\theta_T\,\hat{\mathbf{x}} + \cos\theta_T\,\hat{\mathbf{y}})\end{aligned}\right\} \tag{1.3}$$

Since there is no component of these waves perpendicular to the interface, there are two boundary conditions to meet.

$$\begin{aligned}(i)\quad & \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}\\ (ii)\quad & \tfrac{1}{\mu_1}\vec{B}_1^{\parallel} = \tfrac{1}{\mu_2}\vec{B}_2^{\parallel}\end{aligned} \tag{1.4}$$

Consider the y-component of the first condition and the x-component of the second at $z = 0$ and all times.

$$\tilde{E}_{0_R}\sin\theta_R = \tilde{E}_{0_T}\sin\theta_T \tag{1.5}$$

$$\tilde{E}_{0_R}\sin\theta_R = -\beta\tilde{E}_{0_T}\sin\theta_T, \qquad \beta = \frac{\mu_1 v_1}{\mu_2 v_2} \tag{1.6}$$

The only way these two equations can be simultaneously satisfied with non-zero $\tilde{E}_{0_R}$ and $\tilde{E}_{0_T}$ is if $\theta_R = \theta_T = 0$.

2. Why the H?!

Often, $\vec{H}$ is used in discussing electromagnetic waves. There are two practical reasons for doing so: (1) $\vec{E}\times\vec{H}$ is the power density, and (2) E/H is an impedance.

a) Show that if $E_x(z,t)$ is the standing wave

$$E_x = A\cos\omega t\cos kz,$$

then $H_y(z,t)$ is the standing wave

$$(A/Z)\sin\omega t\sin kz,$$

where $Z = \sqrt{\mu_0/\varepsilon_0}$.

b) Find the electric and magnetic energy densities and the Poynting vector as a function of space and time.

Consider a region of length $\frac{1}{4}\lambda$ extending from a node in E_x to an antinode in E_x. Sketch a plot of E_x and H_y versus z over that region at the times $t = 0, T/8$ and $T/4$.

c) Sketch a plot of the electric energy density, the magnetic energy density, and the total energy density over that region for the same times.

d) Give the direction and magnitude of the Poynting vector $\vec{S}_z$ for those same times.

SOLUTION:

a) It is clear that these are standing waves. We just need to check that the source-free Maxwell's equations are satisified. It is easy to see that the 'divergence' equations are satisfied. We also need to remember the dispersion relation for EM waves in free space.

$$\frac{\omega}{k} = c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu_0\frac{\partial\vec{H}}{\partial t} \tag{2.1}$$

$$\vec{\nabla}\times\vec{E} = \frac{\partial E_x}{\partial z}\,\hat{\mathbf{y}} = -kA\cos\omega t\sin kz\,\hat{\mathbf{y}} \tag{2.2}$$

$$\frac{\partial H_y}{\partial t} = \mu_0 kA\cos\omega t\sin kz \tag{2.3}$$

$$\Rightarrow H_y = \mu_0\frac{k}{\omega}A\sin\omega t\sin kz = (A/Z)\sin\omega t\sin kz \tag{2.4}$$

$$Z = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

It is also quick to check that $\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t}$.

b)

$$\boxed{u_E = \frac{\varepsilon_0}{2}E^2 = \frac{\varepsilon_0}{2}A^2\cos^2\omega t\cos^2 kz} \tag{2.5}$$

$$\boxed{u_B = \frac{1}{2\mu_0}B^2 = \frac{\mu_0}{2}H^2 = \frac{\varepsilon_0}{2}A^2\sin^2\omega t\sin^2 kz} \tag{2.6}$$

$$\vec{S} = \vec{E}\times\vec{H} = \frac{A^2}{Z}\sin\omega t\cos\omega t\sin kz\cos kz\,\hat{\mathbf{z}} \tag{2.7}$$

$$\boxed{\vec{S} = \frac{A^2}{4Z}\sin 2\omega t\sin 2kz\,\hat{\mathbf{z}}} \tag{2.8}$$

See Figure **??**.

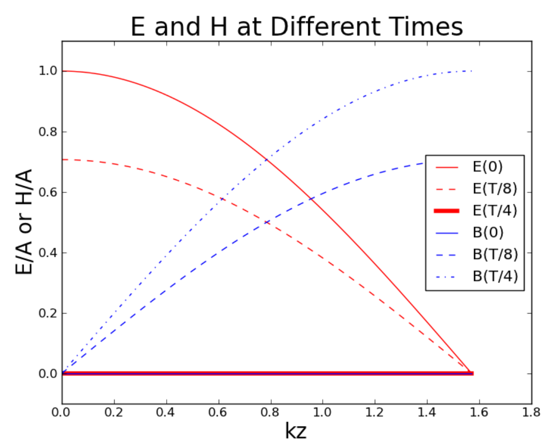

Figure 1: E_x and H_y over a quarter wavelength at different times.

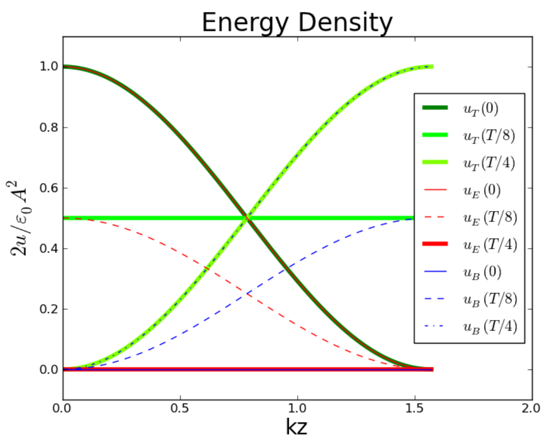

Figure 2: Energy density over a quarter wavelength

c) See Figure **??**

d) In this spatial range and range of time, the Poynting vector is in the positive z direction. Its magnitude at $t = 0$ is 0, its magnitude at $t = T/8$ is $S = A^2/4Z$, and its magnitude at $t = T/4$ is 0 again.

3. Lucite prism

Imagine a Lucite prism (n=1.5) whose cross section is a quarter circle of radius a. As shown in the figure, one flat side rests on the table, and light is incident normal to the other side. The region from P to Q on the table is not illuminated by any light from the prism. Find the distance from the origin O to the point Q.

SOLUTION: Draw a line from Q tangent to the circle. Call the tangent point T. Note that $\angle QOT$ is the same as the angle of incidence of the ray that exits the prism at point T. The rays below this ray exit the prism, bend toward the x-axis and cross it some point further than Q. The rays above this ray reflect internally and cross the x-axis inside the prism. Thus $\angle QOT$ is the critical angle for total internal reflection. $\sin\theta_c = 1/n$.

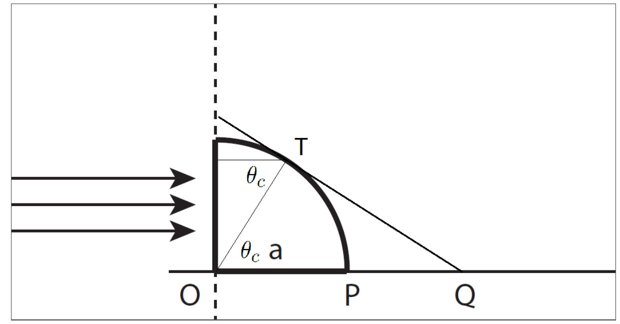

Figure 3: Problem 3 figure.

We can see from the geometry that the distance $x = OQ$ is given by:

$$x = a\sec\theta_c = \frac{a}{\cos\theta_c} = \frac{a}{\sqrt{1-\sin^2\theta_c}} \tag{3.1}$$

$$\boxed{x = \frac{a}{\sqrt{1-(1/1.5)^2}} \approx 1.34a} \tag{3.2}$$

4. Reflection and transmission with a linear dielectric

You are building a microwave tower, and wish to protect the antenna from the nasty upstate NY winter weather. Your antenna broadcasts at $\nu = \omega/2\pi = 10$ GHz, and you will construct your shielding from plastic with an index of refraction $n = 2.5$. What is the *minimum* (nonzero) thickness of plastic required to obtain 100% transmission? Assume that the microwave radiation will be normally incident on the shield. **Important note:** You can *not* solve this problem by applying the single boundary results we found in class. You *must* consider the $\vec{E}$ and $\vec{B}$ boundary conditions at both boundaries *simultaneously.*

SOLUTION: First the incident wave,

$$\vec{E}_1 = E_1^0 \hat{\mathbf{x}}\, e^{i(kz-\omega t)}, \quad \vec{B}_1 = E_1^0 \hat{\mathbf{y}}\, e^{i(kz-\omega t)},$$

where we've used that in air $n = 1$ and $B_1^0 = E_1^0$. And here we have $k = \omega/c$. We want the 100% transmission condition, so we won't write a reflected wave in air.

Next the wave in plastic traveling in the incident direction

$$\vec{E}_2 = E_2^0 \hat{\mathbf{x}}\, e^{i(k'z-\omega t)}, \quad \vec{B}_2 = nE_2^0 \hat{\mathbf{y}}\, e^{i(k'z-\omega t)},$$

where $k' = n\omega/c$ and $n = 2.5$. Then the reflected wave in the plastic

$$\vec{E}_2' = -E_2'^0 \hat{\mathbf{x}}\, e^{i(-k'z-\omega t)}, \quad \vec{B}_2' = nE_2'^0 \hat{\mathbf{y}}\, e^{i(-k'z-\omega t)}.$$

And finally the transmitted wave in air,

$$\vec{E}_3 = E_3^0 \hat{\mathbf{x}}\, e^{i(kz-\omega t)}, \quad \vec{B}_3 = E_3^0 \hat{\mathbf{y}}\, e^{i(kz-\omega t)}.$$

The boundary conditions we have at $E_\parallel$ and $B_\parallel$ are continuous at both $z = 0$ and $z = a$ where a is the width of the plastic. We get four equations,

$$\begin{aligned} E_1^0 &= E_2^0 - E_2'^0 \\ E_1^0 &= n(E_2^0 + E_2'^0) \\ E_3^0 e^{ika} &= E_2^0 e^{ik'a} - E_2'^0 e^{-ik'a} \\ E_3^0 e^{ika} &= n(E_2^0 e^{ik'a} + E_2'^0 e^{-ik'a}). \end{aligned}$$

Combine 1st and 2nd, and combine 3rd and 4th, we get two equations,

$$\begin{aligned} E_2^0 - E_2'^0 &= n(E_2^0 + E_2'^0) \\ E_2^0 e^{ik'a} - E_2'^0 e^{-ik'a} &= n(E_2^0 e^{ik'a} + E_2'^0 e^{-ik'a}) \end{aligned}$$

Multiply 1st by $e^{ik'a}$ and subtract the 2nd, we get

$$-E_2'^0(e^{ik'a} - e^{-ik'a}) = nE_2'^0(e^{ik'a} - e^{-ik'a}) \Rightarrow (n+1)E_2'^0(e^{ik'a} - e^{-ik'a}) = 0,$$

This means $\sin k'a = 0$, so $k'a = m\pi$ where $m = 0, 1, 2, \ldots$ For the smallest non-zero thickness, we have

$$a = \frac{\pi}{k'} = \frac{\pi}{n\omega/c} = 6\,\text{mm},$$

where we've used $n = 2.5$ and $\omega/2\pi = 10^{10}$ Hz.

5. Getting dizzy?

Show that

$$\vec{E} = (\hat{x} + i\hat{y})E_0 e^{i(\omega t - kz)}, \quad \vec{H} = (-i\hat{x} + \hat{y})\frac{E_0}{Z} e^{i(\omega t - kz)}$$

represents a circularly polarized wave. If you watch the time variation of the electric field at a fixed position, will the direction of the field rotate in the right- or left-handed sense with respect to the direction of travel ($+z$)? If you could take a snapshot of the electric field over space, in which sense would the direction rotate? Repeat these questions for the magnetic field.

SOLUTION: First of all, we conclude from inspection that $\vec{E}$ and $\vec{H}$ obey the wave equation. We just need to check Maxwell's equations and show that the wave is circularly polarized. Maxwells equation:

1.
$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 \tag{5.1}$$

2.
$$\vec{\nabla} \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \tag{5.2}$$

3. $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ works if $\omega/k = v = Z/\mu$.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial E_y}{\partial z}\,\hat{\mathbf{x}} + \frac{\partial E_x}{\partial z}\,\hat{\mathbf{y}} = (-\hat{\mathbf{x}} - i\hat{\mathbf{y}})kE_0 e^{i(\omega t - kz)} \tag{5.3}$$

$$-\frac{\partial \vec{B}}{\partial t} = -\mu\frac{\partial \vec{B}}{\partial t} = -\mu(\hat{\mathbf{x}} + i\hat{\mathbf{y}})\omega(E_0/Z)e^{i(\omega t - kz)} \tag{5.4}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{5.5}$$

4.
$$\vec{\nabla} \times \vec{H} = (i\hat{\mathbf{x}} - \hat{\mathbf{y}})k\frac{E_0}{Z}e^{i(\omega t - kz)} \tag{5.6}$$

$$\frac{\partial \vec{D}}{\partial t} = \varepsilon\frac{\partial \vec{E}}{\partial t} = (i\hat{\mathbf{x}} - \hat{\mathbf{y}})\varepsilon\omega E_0 e^{i(\omega t - kz)} \tag{5.7}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \quad \text{if} \quad \omega/k = v = 1/\varepsilon Z \tag{5.8}$$

The last two Maxwell's equations are satisfied if $Z = \sqrt{\mu/\varepsilon}$ and $v = 1/\sqrt{\varepsilon\mu}$.

Now we need to show that the wave is circularly polarized. We take the real part of the electric field.

$$Re[\vec{E}] = E_0\left(\cos(\omega t - kz)\,\hat{\mathbf{x}} - \sin(\omega t - kz)\,\hat{\mathbf{y}}\right) \tag{5.9}$$

We see that this is the parametrization of a circle. Similarly,

$$Re[\vec{B}] = \frac{E_0}{Z}(\sin(\omega t - kz)\,\hat{\mathbf{x}} + \cos(\omega t - kz)\,\hat{\mathbf{y}}). \tag{5.10}$$

A convenient position to look at is $z = 0$ The electric field there has the form

$$\vec{E}(0,t) = E_0(\cos\omega t\,\hat{\mathbf{x}} - \sin\omega t\,\hat{\mathbf{y}}) \tag{5.11}$$

This is a vector that rotates in a left handed sense to $+z$ as time increases. Now consider a snapshot of the wave at time $t = 0$.

$$\vec{E}(z,0) = E_0(\cos kz\,\hat{\mathbf{x}} + \sin kz\,\hat{\mathbf{y}}) \tag{5.12}$$

This is a vector that rotates in a right handed sense to $+z$ as z increases.

Since $\vec{H} = -\frac{i}{Z}\vec{E}$, the magnetic field rotates in the same sense as the electric field.

6. Field in a box

Show that the electromagnetic fields described by

$$\vec{E} = E_0\hat{z}\cos kx\cos ky\cos\omega t$$

and

$$\vec{H} = H_0(\cos kx\sin ky\hat{x} - \sin kx\cos ky\hat{y})\sin\omega t$$

will satisfy Maxwell's equations for a non absorbing, nonmagnetic media with index of refraction n if $E_0 = \alpha H_0$ and $\omega = \beta k$. Determine α and β. These fields can exist in a dielectric enclosed by a square metal box of dimensions π/k in the x and y directions and of arbitrary height. What do the E- and H-fields look like inside the box? Make a sketch of some field lines.

SOLUTION: Maxwell's equations:

1.
$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_z}{\partial z} = 0 \tag{6.1}$$

2.
$$\vec{\nabla} \cdot \mu \vec{H} = \mu \frac{\partial H_x}{\partial x} + \mu \frac{\partial H_y}{\partial y} \tag{6.2}$$
$$= \mu H_0 k(-\sin kx \sin ky + \sin kx \sin ky) \sin \omega t = 0 \tag{6.3}$$

3.
$$\vec{\nabla} \times \vec{E} = \frac{\partial E_z}{\partial y} \hat{\mathbf{x}} - \frac{\partial E_z}{\partial x} \hat{\mathbf{y}} \tag{6.4}$$
$$= -E_0 k(\cos kx \sin ky\, \hat{\mathbf{x}} - \sin kx \cos ky\, \hat{\mathbf{y}}) \cos \omega t \tag{6.5}$$
$$-\frac{\partial \vec{B}}{\partial t} = -\mu H_0 \omega(\cos kx \sin ky\, \hat{\mathbf{x}} - \sin kx \cos ky\, \hat{\mathbf{y}}) \cos \omega t \tag{6.6}$$
$$\boxed{\Rightarrow E_0 k = \mu \omega H_0} \tag{6.7}$$

4.
$$\vec{\nabla} \times \vec{H} = \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \hat{\mathbf{z}} \tag{6.8}$$
$$= -2 H_0 k \cos kx \cos ky \sin \omega t\, \hat{\mathbf{z}} \tag{6.9}$$
$$\frac{\partial \vec{D}}{\partial t} = -\varepsilon E_0 \omega \cos kx \cos ky \sin \omega t\, \hat{\mathbf{z}} \tag{6.10}$$
$$\boxed{\Rightarrow \varepsilon \omega E_0 = 2 k H_0} \tag{6.11}$$

The last two Maxwell's equations define a pair of equations for α and β.

$$\begin{array}{l} \alpha = \mu \beta \\ \alpha = \frac{2}{\varepsilon \beta} \end{array} \tag{6.12}$$

$$\Rightarrow \quad \begin{cases} \alpha = \sqrt{2 \frac{\mu}{\varepsilon}} \\ \beta = \sqrt{\frac{2}{\varepsilon \mu}} \end{cases} \tag{6.13}$$

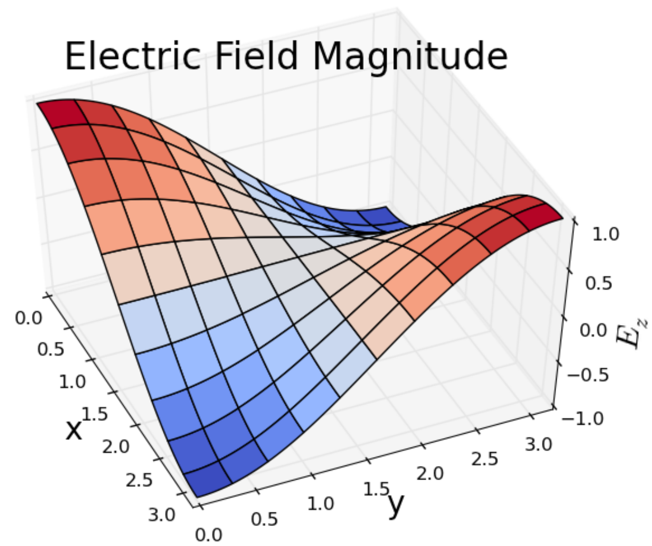

Figure 4: Electric Field Magnitude (z-direction)

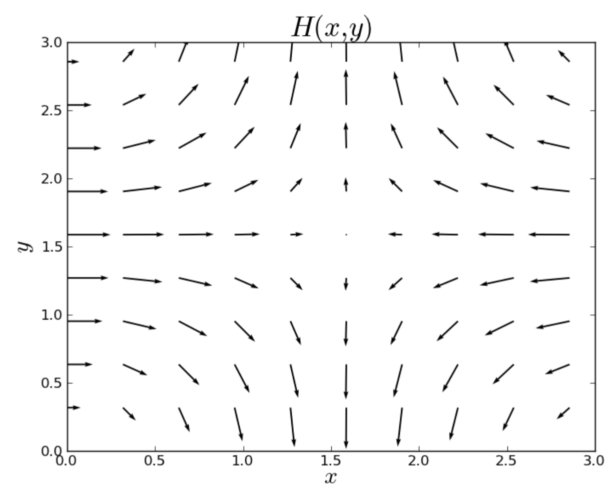

Figure 5: Magnetic Field Vectors

7. Reflections on transmission

Griffiths 9.14 (3rd ed. 9.13). Calculate the *exact* reflection and transmission coefficients, *without* assuming $\mu_1 = \mu_2 = \mu_0$. Confirm that $R + T = 1$.

SOLUTION: Note from Griffiths equations 9.79 and 9.81 that μ and n only show up in the combination n/μ. Thus if we replace each n in equations 9.86 and 9.87 with n/μ, we will have the results we want.

$$R = \left(\frac{n_1/\mu_1 - n_2/\mu_2}{n_1/\mu_1 + n_2/\mu_2}\right)^2 \tag{7.1}$$

$$T = \frac{4n_1n_2/\mu_1\mu_2}{(n_1/\mu_1 + n_2/\mu_2)^2} \tag{7.2}$$

$$R + T = \frac{n_1^2/\mu_1^2 + n_2^2/\mu_2^2 - 2n_1n_2/\mu_1\mu_2 + 4n_1n_2/\mu_1\mu_2}{(n_1/\mu_1 + n_2/\mu_2)^2} \tag{7.3}$$

$$R + T = \frac{(n_1/\mu_1 + n_2/\mu_2)^2}{(n_1/\mu_1 + n_2/\mu_2)^2} = 1 \tag{7.4}$$

8. Don't get phased

Griffiths 9.16 (3rd ed.: 9.15). Suppose $Ae^{iax} + Be^{icx} = Ce^{icx}$, for some nonzero constants A, B, C, a, b, c, and for all x. Prove that $a = b = c$ and $A + B = C$.

SOLUTION: First note that $Ae^{iax} + Be^{ibx} = Ce^{icx}$ must be true for *all* x; in particular $x = 0$ implies $A + B = C$. Also note that this means the first and second derivatives of the equation should also be true at $x = 0$. This gives the following set of equations.

$$\begin{aligned} A + B &= C \\ Aa + Bb &= Cc \\ Aa^2 + Bb^2 &= Cc^2 \end{aligned} \tag{8.1}$$

If we eliminate C with the first from the second and third we get:

$$\begin{aligned} Aa + Bb &= (A + B)c \\ Aa^2 + Bb^2 &= (A + B)c^2 \end{aligned} \tag{8.2}$$

Eliminating c from these gives

$$\begin{aligned} & (A + B)(Aa^2 + Bb^2) = (Aa + Bb)^2 \\ \Rightarrow\quad & A^2a^2 + B^2b^2 + AB(a^2 + b^2) = A^2a^2 + B^2b^2 + 2ABab \\ \Rightarrow\quad & AB(a^2 + b^2 - 2ab) = 0 \\ \Rightarrow\quad & (a - b)^2 = 0 \quad \Rightarrow \quad a = b \end{aligned} \tag{8.3}$$

Plugging this back into the second of Eq. (**??**) and subtracting c times the first of Eq. (**??**) gives:

$$(A + B)(a - c) = 0 \quad \Rightarrow \quad a = c \tag{8.4}$$

Thus we have $A + B = C$ and $a = b = c$.

9. Energy in a conductor

Griffiths 9.21 (3rd ed.: 9.20).

1. Calculate the (time-averaged) energy density of an electromagnetic plane wave in a conducting medium (Eq. 9.138). Show that the magnetic contribution always dominates. $[Answer: \ (k^2/2\mu\omega^2)E_0^2 e^{-2\kappa z}]$
2. Show that the intensity is $(\kappa/2\mu\omega)E_0^2 e^{-\kappa z}$.

SOLUTION: Throughout this problem, the notation of Griffiths section 9.4 will be used. Griffiths Equation 9.138 gives the form of the EM waves in a conductor.

$$\left.\begin{aligned}\vec{E}(z,t) &= E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E)\,\hat{\mathbf{x}}\\ \vec{B}(z,t) &= B_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E+\phi)\,\hat{\mathbf{y}}\end{aligned}\right\} \tag{9.1}$$

The following relations from Griffiths will also be useful.

$$k = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{1/2}$$

$$\kappa = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{1/2}$$

$$K = \sqrt{k^2+\kappa^2} = \omega\sqrt{\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}}$$

$$\tan\phi = \frac{\kappa}{k} \qquad B_0 = E_0\frac{K}{\omega}$$

a) Let u_E be the energy density of the electric field and $\langle u_E\rangle$ denote its time average.

$$u_E = \frac{\varepsilon}{2}E_0^2 e^{-2\kappa z}\cos^2(kz-\omega t+\delta_E) \tag{9.2}$$

Now the time average of $\cos^2(\omega t)$ is $\frac{1}{2}$.

$$\boxed{\langle u_E\rangle = \frac{\varepsilon}{4}E_0^2 e^{-2\kappa z}} \tag{9.3}$$

$$u_B = \frac{1}{2\mu}B_0^2 e^{-2\kappa z}\cos^2(kz-\omega t+\delta_E+\phi) \tag{9.4}$$

$$\boxed{\langle u_B\rangle = \frac{1}{4\mu}B_0^2 e^{-2\kappa z} = \frac{\varepsilon}{4}E_0^2\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\,e^{-2\kappa z}} \tag{9.5}$$

From this we can see that the contribution to the energy density from the magnetic field is larger than the contribution from the electric field since $\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} > 1$.

The total energy density is found by adding the two.

$$\langle u\rangle = \langle u_B\rangle + \langle u_E\rangle = \frac{\varepsilon}{4}E_0^2\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]e^{-2\kappa z} \tag{9.6}$$

Now we note that from the definition of k

$$\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1 = \frac{2k^2}{\varepsilon\mu\omega^2} \tag{9.7}$$

$$\boxed{\langle u\rangle = \frac{k^2}{2\mu\omega^2}E_0^2 e^{-2\kappa z}} \tag{9.8}$$

b) $I = \langle S\rangle$, so we need to find the Poynting vector.

$$\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B}$$

$$\vec{S} = \frac{1}{\mu}E_0B_0\cos(kz-\omega t+\delta_E)\cos(kz-\omega t+\delta_E+\phi)\,\hat{\mathbf{z}} \tag{9.9}$$

We can rewrite the second trig function as

$$\cos(kz-\omega t+\delta_E+\phi) = \cos(kz-\omega t+\delta_E)\cos\phi - \sin(kz-\omega t+\delta_E)\sin\phi \tag{9.10}$$

The second term will average to 0, since $\sin x$ and $\cos x$ integrate to zero over one period.

$$I = \langle\frac{1}{\mu}E_0B_0\cos^2(kz-\omega t+\delta_E)\cos\phi\rangle \tag{9.11}$$

$$I = \frac{1}{2\mu}E_0B_0\cos\phi \tag{9.12}$$

Now we need to get $\cos\phi$ from the expression for $\tan\phi$. This can be done by drawing a right triangle with one angle ϕ, opposite leg κ and adjacent leg k.

$$\cos\phi = \frac{k}{\sqrt{k^2+\kappa^2}} = \frac{k}{K} \tag{9.13}$$

$$B_0\cos\phi = E_0\frac{K}{\omega}\frac{k}{K} = E_0\frac{k}{\omega} \tag{9.14}$$

$$\boxed{I = \frac{k}{2\mu\omega}E_0^2 e^{-2\kappa z}} \tag{9.15}$$

10. Any reception?

Consider a car entering a tunnel that is 15 m wide and 4 m high, and that the walls are good conductors. Determine whether or not AM radio waves will propagate in the tunnel.

SOLUTION: The tunnel is a waveguide. The lowest cuttoff frequency is for the TE10 mode. Then $\omega_{10} = c\pi/a$ where a is the larger dimension of the guide.

$$f_c = \frac{\omega_{10}}{2\pi} = \frac{c}{2a} = \frac{3\times10^8}{2\times15} = 1\times10^7 = 10\text{MHz}$$

AM radio frequencies are sub MHz. The AM waves will not propagate.

11. Just one mode, please

Griffiths 9.29 (3rd ed.: 9.28). Consider a rectangular wave guide with dimensions 2.28 cm × 1.01 cm. What TE modes will propagate in this wave guide, if the driving frequency is 1.70×10^{10} Hz? Suppose you wanted to excite only *one* TE mode; what range of frequencies could you use? What are the corresponding wavelengths (in open space)?

SOLUTION:

$$f_{mn} = \frac{\omega_{mn}}{2\pi} = \frac{c}{2}\sqrt{(m/a)^2+(n/b)^2}$$

$$\begin{aligned}
f_{10} &= = 6.58\times10^9 \to \lambda = c/f = 4.56\text{cm}\\
f_{01} &= 1.49\times10^{10} \to= 2.02\text{cm}\\
f_{11} &= 1.62\times10^{10} \to= 1.85\text{cm}\\
f_{20} &= 1.32\times10^{10} \to= 2.28\text{cm}\\
f_{02} &= 2.97\times10^{10} \to= 1.01\text{cm}\\
f_{12} &= 3.04\times10^{10} \to= 0.99\text{cm}\\
f_{21} &= 1.98\times10^{10} \to= 1.5\text{cm}\\
f_{30} &= 1.97\times10^{10} \to= 1.52\text{cm}\\
f_{31} &= 2.47\times10^{10} \to= 1.2\text{cm}
\end{aligned}$$

The cutoff frequencies for $T_{10}, T_{01}, T_{11}, T_{20}$ are all below 1.7×10^{10} so the driving frequency will propagate in these TE modes (but no others). To excite only one TE mode, the driving frequency should be in the range $6.58\times10^9 < f < 1.32\times10^{10}$ and $4.56\text{cm} > \lambda > 2.28\text{cm}$.