Physics 3323, Fall 2016 Problem Set 2 *due Sep 9, 2016*

1. What's my charge?

A spherical region of radius R is filled with a charge distribution that gives rise to an electric field inside of the form $\vec{E} = (E_0/R^2)r\vec{r}$, where $\vec{r}$ is the radius vector drawn for the center of the region, and E_0 is a constant. Find the charge density inside the region.

SOLUTION:
Here we use Gauss's law in differential form.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{1.1}$$

$$\vec{E} = (E_0/R^2)r\vec{r} = \frac{E_0 r^2}{R^2}\hat{\mathbf{r}}$$

$$\rho = \epsilon_0 \left[\vec{\nabla} \cdot \left(E_0 \frac{r^2}{R^2}\hat{\mathbf{r}}\right)\right] = \frac{\epsilon_0 E_0}{R^2}\frac{1}{r^2}\frac{\partial}{\partial r}r^4 = \frac{4\epsilon_0 E_0}{R^2} r \tag{1.2}$$

2. Electric Field of Coaxial Cable

A long coaxial cable carries a uniform volume charge density ρ throughout its solid inner cylinder of radius a, and a uniform surface charge density σ on its thin outer cylinder of radius b. The cylinders are concentric and the cable is overall electrically neutral.

a) Find the electric field $\vec{E}$ everywhere in space.

b) Sketch the field.

c) Sketch the magnitude of the field as a function of the distance from the cylinders' center.

SOLUTION:
The first thing to note is that for the cable to be electrically neutral we need $V_{in}\rho + A_{out}\sigma = 0$ wher V_{in} is the volume of the inner cylinder and A_{out} is the surface area of the outer cylinder. This leads to

$$\pi a^2 L\rho + 2\pi bL\sigma = 0 \ \Rightarrow \ \sigma = -\frac{a^2}{2b}\rho \tag{2.1}$$

a)By symmetry for a long cable the electric field should be in the radial ($\hat{s}$) direction and can only depend on s. We will imagine cylindrical Gaussian surfaces of various radii whose axes coincide with the axis of the cable. Then $\vec{E}$ is parallel to $d\vec{A}$ so that $\vec{E} \cdot d\vec{A} = E\,dA$. The ends of the surface will not contribute since $\vec{E} \cdot d\vec{A} = 0$ there.

$$\frac{1}{\epsilon_0}\int \rho\,dV = \int \vec{E} \cdot d\vec{A} = E2\pi sL$$

$$E = \frac{1}{2\pi\epsilon_0 sL}\int \rho\,dV \tag{2.2}$$

We need to consider three different regions: the region where $s < a$, where $a < s < b$ and where $s > b$.
For $s < a$:

$$\int \rho\,dV = \pi s^2 L\rho$$

$$\vec{E} = \frac{s\rho}{2\epsilon_0}\hat{s} \tag{2.3}$$

For $a < s < b$:

$$\int \rho\,dV = \pi a^2 L\rho$$

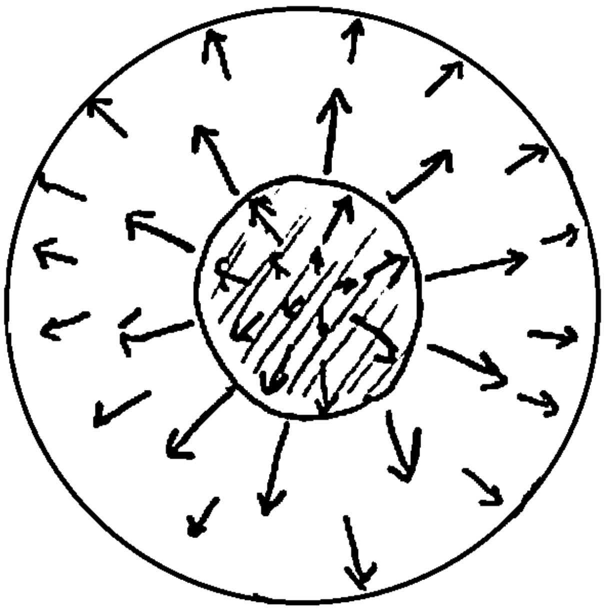
$$\vec{E} = \frac{a^2\rho}{2s\epsilon_0}\hat{s} \tag{2.4}$$

For $a > b$:

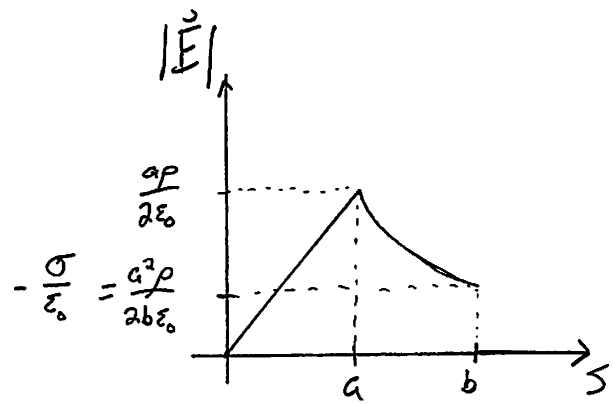
$$\int \rho \, dV = \pi a^2 L\rho + 2\pi b\sigma L = 0$$

$$\vec{E} = 0 \tag{2.5}$$

b)



c)



3. Gradients of 1/r

Demonstrate that

$$\vec{\nabla}_{\vec{r}} \frac{1}{|\vec{r} - \vec{r_1}|} = -\vec{\nabla}_{\vec{r_1}} \frac{1}{|\vec{r} - \vec{r_1}|} = -\frac{\vec{r} - \vec{r_1}}{|\vec{r} - \vec{r_1}|^3} \tag{3.1}$$

SOLUTION:

The most straight-forward way of doing this is to write both of the first two in terms of the coordinates and show they both are equal to the third.

a)

$$\begin{aligned} \vec{\nabla}_{\vec{r}} \frac{1}{|\vec{r} - \vec{r_1}|} &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \frac{1}{\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z1)^2}} \\ &= -\frac{2(x-x_1)\hat{\mathbf{i}}+2(y-y_1)\hat{\mathbf{j}}+2(z-z_1)\hat{\mathbf{k}}}{2((x-x_1)^2+(y-y_1)^2+(z-z1)^2)^{3/2}} \\ &= -\frac{\vec{r}-\vec{r_1}}{|\vec{r}-\vec{r_1}|^3} \end{aligned} \tag{3.2}$$

b)

$$-\vec{\nabla}_{\vec{r_1}} \frac{1}{|\vec{r} - \vec{r_1}|} = -\left(\hat{\mathbf{i}}\frac{\partial}{\partial x_1} + \hat{\mathbf{j}}\frac{\partial}{\partial y_1} + \hat{\mathbf{k}}\frac{\partial}{\partial z_1}\right) \frac{1}{\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z1)^2}}$$

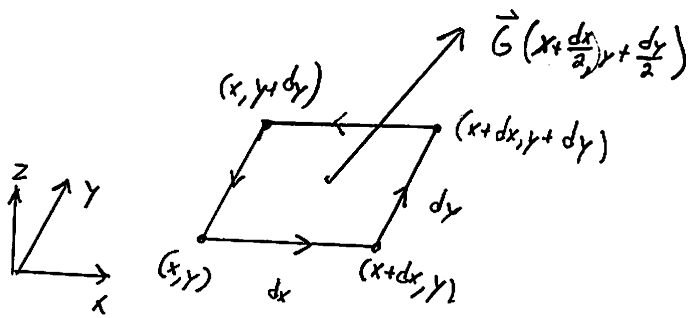
$$= \frac{-2(x-x_1)\hat{\mathbf{i}}-2(y-y_1)\hat{\mathbf{j}}-2(z-z_1)\hat{\mathbf{k}}}{2((x-x_1)^2+(y-y_1)^2+(z-z1)^2)^{3/2}}$$
$$= -\frac{\vec{r}-\vec{r_1}}{|\vec{r}-\vec{r_1}|^3} \tag{3.3}$$

4. Makes your hair curl

By examining the line integral of $\vec{G}$ around a small loop in the xy plane with diagonal corners at (x, y) and $(x + dx, y + dy)$, prove the standard form for curl$\vec{G} = \vec{\nabla} \times \vec{G}$ in Cartesian coordinates. Consider what component this line integral allows you to evaluate, and use symmetry to argue the forms for the other components.

SOLUTION: The line integral of $\vec{G}$ around this loop should be the same as the z-component of the curl of $\vec{G}$ multiplied by the area of the loop. This is just Stokes theorem.

$$\oint \vec{G} \cdot d\vec{l} = \int \left(\vec{\nabla} \times \vec{G}\right) \cdot d\vec{A} \tag{4.1}$$



First we work with the left hand side. Constructing the rectangular loop in the xy-plane, we see there are four line segments to add up. For small dx, dy and smooth $\vec{G}$ we can just multiply the value of the parallel component of $\vec{G}$ at the midpoint of the line by the length of the line segment. This approximation becomes exact in the limit $dx, dy \to 0$. We then have for the left hand side of Eq. (**??**):

$$G_x\left(x+\frac{dx}{2}, y\right) \quad +G_y\left(x+dx, y+\tfrac{dy}{2}\right)$$
$$-G_x\left(x+\tfrac{dx}{2}, y+dy\right)$$
$$-G_y\left(x, y+\tfrac{dy}{2}\right)$$

$$= \left[G_x\left(x+\tfrac{dx}{2}, y\right) - G_x\left(x+\tfrac{dx}{2}, y+dy\right)\right] dx$$
$$+\left[G_y\left(x+dx, y+\tfrac{dy}{2}\right) - G_y\left(x, y+\tfrac{dy}{2}\right)\right] dy$$
$$\approx \left[-\frac{\partial G_x}{\partial y}dy\right] dx + \left[\frac{\partial G_y}{\partial x}dx\right] dy = \left[\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}\right] dxdy \tag{4.2}$$

For the right hand side of Eq. (**??**), we approximate the function $\vec{G}$ as constant over the area. (If you like, use the midpoint of the area to match the derivatives which were calculated at the midpoints of the lines. In the limit $dx, dy \to 0$ these details don't matter). Thus the right hand size is

$$\left(\vec{\nabla} \times \vec{G}\right)_z dxdy \tag{4.3}$$

Equating the two we have

$$\left(\vec{\nabla} \times \vec{G}\right)_z = \frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y} \tag{4.4}$$

The other components of $\left(\vec{\nabla} \times \vec{G}\right)$ can be found in the same manner by cyclic permutation $x \to y \to z \to x$ etc.

5. Div and Curl

Consider the field $\vec{E} = (2x^2 - 2xy - 2y^2)\hat{x} + (-x^2 - 4xy + y^2)\hat{y}$.

a) Is it irrotational? If so, what is the potential function?

b) **C**alculate $\vec{\nabla} \cdot \vec{E}$.

SOLUTION:

a)

$$\vec{\nabla}\times\vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial y}\right)\hat{\mathbf{j}} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{\mathbf{k}}$$

$$= 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + (-2x - 4y) - (-2x - 4y)\hat{\mathbf{k}} = \vec{0}$$

The vector field is irrotational. The potential V can be found by integrating the components of $\vec{E}$ since $E_x = -\frac{\partial V}{\partial x}$ etc. Integrating the x component gives

$$V = \int \left[2y^2 + 2xy - 2x^2\right] dx = 2y^2x + x^2y - \frac{2x^3}{3} + f(y).$$

Here $f(y)$ is some function of y (analogous to the constant of integration in one-dimensional calculus).
Integrating the y component gives

$$V = \int \left[x^2 + 4xy - y^2\right] dy = x^y + 2xy^2 - \frac{y^3}{3} + g(x).$$

These two results together give the solution

$$V = 2xy^2 + x^2y - \frac{2x^3}{3} - \frac{y^3}{3} \qquad (5.1)$$

b)

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial Ez}{\partial z} = 4x - 2y - 4x - 2y = 0 \qquad (5.2)$$

6. Griffiths 2.47, 4^{th} ed.

Find the net force that the southern hemisphere of a uniformly charged solid sphere exerts on the northern hemsiplere. Express your answer in terms of the radius R and the total charge Q. [*Answer:* $(1/4\pi\epsilon_0)(3Q^2/16R^2)$].

SOLUTION: Use Gauss' law to find the electric field inside the sphere. Set the Gaussian spherical surface concentric with the charged sphere. We find

$$\mathbf{E}(4\pi r^2) = \frac{1}{\epsilon_0}\int_0^r \rho r'^2 dr' \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi$$

$$\mathbf{E} = \frac{\rho}{4\pi r^2 \epsilon_0}\frac{4}{3}\pi r^3 \hat{\mathbf{r}} = \frac{\rho}{3\epsilon_0} r\hat{\mathbf{r}}$$

The force on the charge in the upper half of the sphere (polar angle $0 < \theta < \pi/2$, is all in the vertical direction. The vertical component of the field is

$$E_y = \mathbf{E}(\mathbf{r}) \cdot \hat{\mathbf{z}} = E(r)\cos\theta$$

The force is the product of the vertical component of the field and the charge. The force on an infinitesimal volume of charge at r, θ is

$$dF = E(r)\cos\theta\rho r^2 dr \sin\theta d\theta d\phi$$

where $\rho r^2 dr sin\theta d\theta d\phi = dq$. Then the total force is

$$\int dF = \int_0^R \int_0^{\pi/2} \int_0^{2\pi} E(r)\cos\theta\rho d\tau$$

$$= \int_0^R \int_0^{\pi/2} \int_0^{2\pi} E(r)\cos\theta\rho r^2 dr \sin\theta d\theta d\phi$$

$$F = \int_0^R \int_0^{\pi/2} \int_0^{2\pi} \frac{\rho r}{3\epsilon_0}\cos\theta\rho r^2 dr \sin\theta d\theta d\phi$$

$$
\begin{aligned}
&= \frac{\rho^2 R^4}{4 \cdot 3\epsilon_0} \frac{1}{2}(2\pi) \\
&= \left(\frac{4}{3}\pi R^3 \rho\right)^2 \left(\frac{1}{4\pi\epsilon_0}\right) \frac{3}{16R^2} \\
&= \frac{1}{4\pi\epsilon_0} \frac{3Q^2}{16R^2}
\end{aligned}
$$