Physics 3323, Fall 2016

**Problem Set 3**

**due Sep 16, 2016**

**Reading:** Griffiths 2.5 through 3.1

1. **Potential of uniformly charged sphere**

Find the potential inside and outside a uniformly charged solid sphere whose radius is \( R \) and whose total charge is \( q \). Use infinity as your reference point. Compute the gradient of \( V \) in each region, and check that it yields the correct field. Sketch \( V(r) \).

**SOLUTION:** First, we quickly use Gauss’s law in integral form and the spherical symmetry to calculate the electric field both inside and outside the sphere. We know that \( \vec{E} \) is in the \( \hat{r} \) direction.

\[
4\pi r^2 E_{\text{out}} = \frac{q}{\varepsilon_0} \Rightarrow \vec{E}_{\text{out}} = \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} \hat{r}
\]

\[
4\pi r^2 E_{\text{in}} = \frac{q}{\varepsilon_0} \left( \frac{r}{R} \right)^3 \Rightarrow \vec{E}_{\text{in}} = \frac{1}{4\pi \varepsilon_0} \frac{qr}{R^3} \hat{r}
\]

Now we can integrate from a point at infinity to a point at radius \( r \) to get the potential.

\[
V_{\text{out}} = -\int_{\infty}^{r} E_{\text{out}} \, dr
\]

\[
= -\frac{1}{4\pi \varepsilon_0} q \int_{\infty}^{r} \frac{dr}{r^2}
\]

\[
V_{\text{out}} = -\frac{1}{4\pi \varepsilon_0} \left( \frac{q}{r} \right)_{\infty} = \frac{1}{4\pi \varepsilon_0} \frac{q}{r} \quad (1.1)
\]

\[
V_{\text{in}} = -\int_{\infty}^{R} E_{\text{out}} \, dr - \int_{R}^{r} E_{\text{in}} \, dr
\]

\[
= \frac{1}{4\pi \varepsilon_0} \left[ \frac{q}{R} - \left( \int_{R}^{r} \frac{qr}{R^3} \, dr \right) \right]
\]

\[
= \frac{1}{4\pi \varepsilon_0} \left[ \frac{q}{R} - \left( \frac{qr^2}{2R^3} \right)_{R} \right]
\]

\[
= \frac{1}{4\pi \varepsilon_0} \left[ \frac{q}{R} - \frac{qr^2}{2R^3} + \frac{q}{2R} \right]
\]

\[
V_{\text{in}} = \frac{1}{4\pi \varepsilon_0} \frac{q}{R} \left[ 3 - \frac{1}{2} \left( \frac{r}{R} \right)^2 \right] \quad (1.2)
\]

Now we can take the gradient to see if these formulas yield the correct electric field. With spherical symmetry, the gradient takes the form

\[
\nabla V(r) = \frac{\partial V}{\partial r} \hat{r}
\]

\[
-\nabla V_{\text{out}} = -\frac{1}{4\pi \varepsilon_0} \hat{r} \frac{\partial}{\partial r} \left[ \frac{q}{r} \right]
\]
2. Electrostatic energy of a nucleus

Suppose you model the nucleus as a uniformly charged sphere with a total charge \( Q = Ze \) and radius \( R = 1.2 \times 10^{-15} A^{1/3} \) m.

a) Show that the electrostatic energy of such a sphere is given by \( \frac{3Q^2}{20 \pi \varepsilon_0 R} \).

b) Using (a), compute the electrostatic energy of an atomic nucleus, expressing your result in MeV \( \times \frac{Z^2}{A^{1/3}} \).

c) Calculate the change in electrostatic energy when a uranium nucleus (\( Z = 92, A = 238 \)) fissions into two equal fragments.

SOLUTION:

We can use the electric fields calculated in problem 2 to calculate the integral for electrostatic energy.

\[
W = \frac{\varepsilon_0}{2} \int E^2 dV
= \frac{\varepsilon_0}{2} \int_0^R E_{\text{in}}^2 4\pi r^2 dr + \frac{\varepsilon_0}{2} \int_{R}^{\infty} E_{\text{out}}^2 4\pi r^2 dr
= \frac{4\pi Q^2}{32\pi^2 \varepsilon_0} \left[ \int_0^R \frac{r^4}{R^6} dr + \int_{R}^{\infty} \frac{1}{r^2} dr \right]
\]

b) \[
W = \frac{3(Ze)^2}{20 \pi \varepsilon_0 (1.2 \times 10^{-15} \text{ m}) A^{1/3}}
= \frac{3(1.602 \times 10^{-19} \text{ C})^2}{20 \pi \cdot 8.854 \times 10^{-12} \text{ F/m} \cdot 1.2 \times 10^{-15} \text{ m}} \times \frac{1 \text{ eV}}{1.602 \times 10^{-19} \text{ J} \cdot A^{1/3}} \times \frac{Z^2}{A^{1/3}}
\]

\[
W = 0.72 \text{ MeV} \times \frac{Z^2}{A^{1/3}}
\]

(2.2)

c) \[
W_{\text{U}_{238}^{92} \rightarrow 2\text{Pd}_{119}^{46}}
\]
\[
\Delta W = 0.72 \text{ MeV} \times \left( \frac{92^2}{238^{1/3}} - 2 \frac{46^2}{119^{1/3}} \right)
\]
\[
\Delta W = 364 \text{ MeV}
\]

(2.3)

3. Screened Coulomb potential

Consider the screened Coulomb potential of a point charge \( q \), which can arise in plasma physics or in analyzing conduction electrons in semiconductors. The potential is given by:

\[
V(\vec{r}) = \frac{q}{4\pi \varepsilon_0} \frac{e^{-r/\lambda}}{r},
\]

(3.1)
where $\lambda$ is the “screening length”, a constant. Such a potential arises because a charged particle will attract oppositely charged particles, or equivalently, repel like-sign charges, which surround the original particle. Such a cloud will mask, or screen, the charge of the original particle, giving rise to a field that dies much more rapidly than the bare Coulomb interaction.

a) Determine the electric field $\vec{E}(\vec{r})$ associated with this potential.

b) Find the charge distribution $\rho(\vec{r})$ associated with this electric field. Careful about the origin (hint: how can you write $\rho(\vec{r})$ for a point charge?) Make a sketch of your result that captures the dominant features.

c) Show that the total charge involved is zero by both integrating $\rho(\vec{r})$ over all space, and by using the integral form of Gauss’ law on the electric field.

---

**SOLUTION:**

a)  

$$
\vec{E} = -\nabla V \\
= -\frac{q}{4\pi\varepsilon_0} \hat{r} \frac{\partial}{\partial r} \left[ \frac{e^{-r/\lambda}}{r} \right] \\
= -\frac{q}{4\pi\varepsilon_0} \hat{r} \left[ -\frac{r}{\lambda} e^{-r/\lambda} \right] \\
\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2} \left[ 1 + \frac{r}{\lambda} \right] e^{-r/\lambda} \hat{r} 
$$

(3.2)

b) Here we can use Gauss’s law in differential form.

$$
\rho = \varepsilon_0 \nabla \cdot \vec{E} 
$$

$$
= \varepsilon_0 \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{q}{4\pi\varepsilon_0 r^2} \left[ 1 + \frac{r}{\lambda} \right] e^{-r/\lambda} \\
= \frac{q}{4\pi r^2} \frac{\partial}{\partial r} \left[ \left(1 + \frac{r}{\lambda} \right) e^{-r/\lambda} \right] \\
\rho = \frac{qe^{-r/\lambda}}{4\pi r^2} 
$$

This result should make us question our method, since it implies a charge density that is everywhere the opposite sign of $q$. But we know that there is a point with charge density with the same sign as $q$, namely the charge itself. The reason our method of taking a derivative fails is because the function is not defined at $r = 0$. We can write the density of the charge at the origin with a delta function

$$
\rho = q\delta^3(\vec{r}) - \frac{qe^{-r/\lambda}}{4\pi r^2}. 
$$

(3.3)

c) First by integrating the charge density over all of space.

$$
Q = \int \rho \, dV = \int dV \left[ q\delta^3(\vec{r}) - \frac{qe^{-r/\lambda}}{4\pi r^2} \right] \\
= q \left[ 1 - \int_0^\infty \frac{e^{-r/\lambda}}{4\pi r^2} 4\pi r^2 \, dr \right] \\
= q \left[ 1 - \int_0^\infty xe^{-x} \, dx \right] = 0
$$

Now by applying Gauss’s law in integral form where $r$ is the radius of a spherical Gaussian surface.

$$
Q = \lim_{r \to \infty} \varepsilon_0 \int \vec{E} \cdot d\vec{A} \\
= \lim_{r \to \infty} \varepsilon_0 \frac{q}{4\pi\varepsilon_0 r^2} \left[ 1 + \frac{r}{\lambda} \right] e^{-r/4\pi r^2} 
$$
4. Coaxial conductors

Calculate the energy per unit length for two very long coaxial cylindrical shells, neglecting edge effects. The inner and outer cylinders have radii \( a \) and \( b \) and uniform linear charge densities \( \lambda \) and \( -\lambda \), respectively. The charge is uniformly distributed around the cylinder surfaces.

**SOLUTION:**

Here the cylindrical symmetry allows us to use cylindrical Gaussian surfaces to calculate the electric field in each of the regions \( s < a \), \( a < s < b \), and \( s > b \). For both \( s < a \) and \( s > b \), there is no net charge enclosed in the Gaussian surface, so \( \vec{E} = 0 \) there. For \( a < s < b \), we have

\[
2\pi s l E = \frac{\lambda l}{\varepsilon_0} \Rightarrow \vec{E} = \frac{\lambda}{2\pi \varepsilon_0 s} \hat{s}
\]

Now we can integrate the energy per unit length.

\[
\frac{W}{l} = \frac{\varepsilon_0}{2l} \int E^2 dV
\]

\[
= \frac{\varepsilon_0}{2l} \int_{a}^{b} \frac{\lambda^2}{4\pi \varepsilon_0 s^2} \cdot 2\pi l s \, ds
\]

\[
= \frac{\lambda^2}{4\pi \varepsilon_0} \int_{a}^{b} \frac{ds}{s} = \frac{\lambda^2}{4\pi \varepsilon_0} \ln \left( \frac{b}{a} \right)
\]

\[
\frac{W}{l} = \frac{\lambda^2}{4\pi \varepsilon_0} \ln \left( \frac{b}{a} \right) \tag{4.1}
\]

5. A spherical capacitor

A capacitor consists of two concentric spherical shells. The inner shell, conductor 1, has radius \( a \), and the outer shell, conductor 2, has radius \( b \). For this two-conductor system, find \( C_{11}, C_{12} \) and \( C_{22} \).

**SOLUTION:** Case 1: set \( V_2 = 0 \) on the outer conductor while applying a voltage \( V_1 \) to the inner conducting sphere. Suppose we have charge \( q_2 \) on the outer conductor, and charge \( q_1 \) on the inner conductor. For the space between the two conductors, Gauss’ law tells us that

\[
\vec{E} = k \frac{q_1}{r^2} \hat{r},
\]

so

\[
V_2 - V_1 = -\int_{a}^{b} \vec{E} \cdot d\vec{l} = k q_1 \left( \frac{1}{b} - \frac{1}{a} \right).
\]

But here, \( V_2 - V_1 = -V_1 \), so

\[
q_1 = 4\pi \varepsilon_0 \frac{ab}{b-a} V_1
\]

and hence

\[
C_{11} = 4\pi \varepsilon_0 \frac{ab}{b-a}.
\]

The charge \( q_2 \) must cancel the potential contribution from \( q_1 \). Since for spherical geometries the electric field and potential both act like charges at the center of the sphere, we see we must have \( q_2 = -q_1 \) and therefore \( C_{21} = -C_{11} \).

Case 2: set \( V_1 = 0 \) on the inner conductor while applying a voltage \( V_2 \) to the outer conducting sphere. From the same argument we started with above, we now have

\[
V_2 - V_1 = V_2 = -\int_{a}^{b} \vec{E} \cdot d\vec{l} = k q_1 \left( \frac{1}{b} - \frac{1}{a} \right),
\]
and we see that $C_{12} = C_{21}$ as expected. For the outer sphere, we now have

$$V_2 = k \frac{q_1 + q_2}{b}.$$ 

Solving for $q_2$

$$q_2 = -\frac{b}{a} q_1.$$