Reading: Griffiths 2.5 through 3.1

1. Potential of uniformly charged sphere

Find the potential inside and outside a uniformly charged solid sphere whose radius is R and whose total charge is q. Use infinity as your reference pont. Compute the gradent of V in each region, and check that it yields the correct field. Sketch $V(r)$.

SOLUTION: First, we quickly use Gauss's law in integral form and the spherical symmetry to calculate the electric field both inside and outside the sphere. We know that $\vec{E}$ is in the $\hat{\mathbf{r}}$ direction.

$$4\pi r^2 E_{out} = \frac{q}{\varepsilon_0} \;\Rightarrow\; \vec{E}_{out} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

$$4\pi r^2 E_{in} = \frac{q}{\varepsilon_0}\left(\frac{r}{R}\right)^3 \quad \vec{E}_{in} = \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}\hat{\mathbf{r}}$$

Now we can integrate from a point at infinity to a point at radius r to get the potential.

$$V_{out} = -\int_\infty^r E_{out}\,dr$$

$$= -\frac{1}{4\pi\varepsilon_0} q \int_\infty^r \frac{dr}{r^2}$$

$$\boxed{V_{out} = -\frac{1}{4\pi\varepsilon_0}\left(-\frac{q}{r}\right)_\infty^r = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}} \tag{1.1}$$

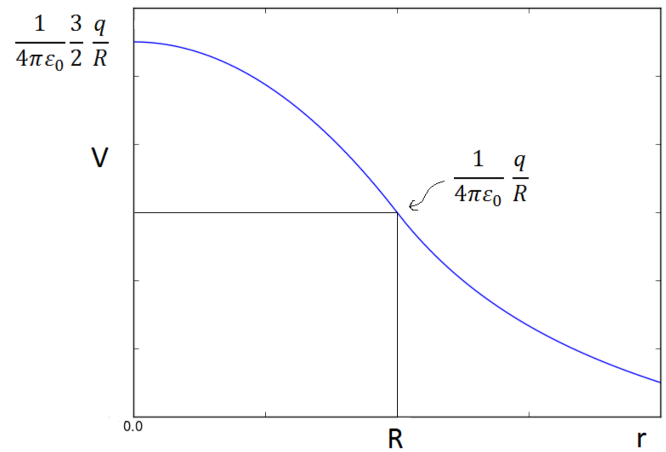
$$V_{in} = -\int_\infty^R E_{out}\,dr - \int_R^r E_{in}\,dr$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{R} - \left(\int_R^r \frac{qr}{R^3}\,dr\right)\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{R} - \left(\frac{qr^2}{2R^3}\right)_R^r\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{R} - \frac{qr^2}{2R^3} + \frac{q}{2R}\right]$$

$$\boxed{V_{in} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}\left[\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2\right]} \tag{1.2}$$



Now we can take the gradient to see if these formulas yield the correct electric field. With spherical symmetry, the gradient takes the form

$$\vec{\nabla} V(r) = \frac{\partial V}{\partial r}\hat{\mathbf{r}}$$

.

$$-\vec{\nabla} V_{out} = -\frac{1}{4\pi\varepsilon_0}\,\hat{\mathbf{r}}\,\frac{\partial}{\partial r}\left[\frac{q}{r}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}} = \vec{E}_{out}$$

$$-\vec{\nabla}V_{in} = -\frac{1}{4\pi\varepsilon_0}\frac{q}{R}\,\hat{\mathbf{r}}\,\frac{\partial}{\partial r}\left[\frac{3}{2} - \frac{1}{2}\frac{r^2}{R^2}\right]$$

$$-\vec{\nabla}V_{in} = \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}\,\hat{\mathbf{r}} = \vec{E}_{in}$$

2. Electrostatic energy of a nucleus

Suppose you model the nucleus as a uniformly charged sphere with a total charge $Q = Ze$ and radius $R = 1.2 \times 10^{-15} A^{1/3}$ m.

a) Show that the electrostatic energy of such a sphere is given $3Q^2/(20\pi\varepsilon_0 R)$.

b) Using (a), compute the electrostatic energy of an atomic nucleus, expressing your result in MeV$\times Z^2/A^{1/3}$.

c) Calculate the change in electrostatic energy when a uranium nucleus ($Z = 92$, $A = 238$) fissions into two equal fragments.

SOLUTION:

a) We can use the electric fields calculated in problem 2 to calculate the integral for electrostatic energy.

$$W = \frac{\varepsilon_0}{2}\int E^2\,dV$$

$$= \frac{\varepsilon_0}{2}\int_0^R E_{in}^2 4\pi r^2\,dr + \frac{\varepsilon_0}{2}\int_R^\infty E_{out}^2 4\pi r^2\,dr$$

$$= \frac{4\pi Q^2}{32\pi^2\varepsilon_0}\left[\int_0^R \frac{r^4}{R^6}\,dr + \int_R^\infty \frac{1}{r^2}\,dr\right]$$

$$= \frac{Q^2}{8\pi\varepsilon_0}\left[\left.\frac{r^5}{5R^6}\right|_0^R - \left.\frac{1}{r}\right|_R^\infty\right]$$

$$= \frac{Q^2}{8\pi\varepsilon_0 R}\left[\frac{1}{5} + 1\right]$$

$$\boxed{W = \frac{3Q^2}{20\pi\varepsilon_0 R}} \tag{2.1}$$

b)

$$W = \frac{3(Ze)^2}{20\pi\varepsilon_0(1.2\times10^{-15}\,m)A^{1/3}}$$

$$= \frac{3(1.602\times10^{-19}\,\mathrm{C})^2}{20\pi\cdot 8.854\times10^{-12}\,\mathrm{F/m}\cdot 1.2\times10^{-15}\,m}\times\frac{1\mathrm{eV}}{1.602\times10^{-19}\mathrm{J}}\times\frac{Z^2}{A^{1/3}}$$

$$\boxed{W = 0.72\,\mathrm{MeV}\times\frac{Z^2}{A^{1/3}}} \tag{2.2}$$

c)

$$\mathrm{U}_{92}^{238} \rightarrow 2\mathrm{Pd}_{46}^{119}$$

$$\Delta W = 0.72\,\mathrm{MeV}\times\left(\frac{92^2}{238^{1/3}} - 2\frac{46^2}{119^{1/3}}\right)$$

$$\boxed{\Delta W = 364\,\mathrm{MeV}} \tag{2.3}$$

3. Screened Coulomb potential

Consider the screened Coulomb potential of a point charge q, which can arise in plasma physics or in analyzing conduction electrons in semiconductors. The potential is given by:

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{e^{-r/\lambda}}{r}, \tag{3.1}$$

where λ is the "screening length", a constant. Such a potential arises because a charged particle will attract oppositely charged particles, or equivalently, repel like-sign charges, which surround the original particle. Such a cloud will mask, or screen, the charge of the original particle, giving rise to a field that dies much more rapidly than the bare Coulomb interaction.

a) Determine the electric field $\vec{E}(\vec{r})$ associated with this potential.

b) Find the charge distribution $\rho(\vec{r})$ associated with this electric field. Careful about the origin (hint: how can you write $\rho(\vec{r})$ for a point charge?) Make a sketch of your result that captures the dominant features.

c) Show that the total charge involved is zero by both integrating $\rho(\vec{r})$ over all space, and by using the integral form of Gauss' law on the electric field.

SOLUTION:

a)

$$\vec{E} = -\vec{\nabla} V$$
$$= -\frac{q}{4\pi\varepsilon_0}\hat{\mathbf{r}}\frac{\partial}{\partial r}\left[\frac{e^{-r/\lambda}}{r}\right]$$
$$= -\frac{q}{4\pi\varepsilon_0}\hat{\mathbf{r}}\left[\frac{-r/\lambda - 1}{r^2}\right]e^{-r/\lambda}$$
$$\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\left[1+\frac{r}{\lambda}\right]e^{-r/\lambda}\,\hat{\mathbf{r}}} \tag{3.2}$$

b) Here we can use Gauss's law in differential form.

$$\rho = \varepsilon_0 \vec{\nabla}\cdot\vec{E}$$
$$= \varepsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r} r^2\left[\frac{q}{4\pi\varepsilon_0 r^2}\left[1+\frac{r}{\lambda}\right]e^{-r/\lambda}\right]$$
$$= \frac{q}{4\pi r^2}\frac{\partial}{\partial r}\left[\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda}\right]$$
$$\rho = -\frac{qe^{-r/\lambda}}{4\pi r\lambda^2}?$$

This result should make us question our method, since it implies a charge density that is everywhere the opposite sign of q. But we know that there is a point with charge density with the same sign as q, namely the charge itself. The reason our method of taking a derivative fails is because the function is not defined at $r = 0$. We can write the density of the charge at the origin with a delta function

$$\boxed{\rho = q\delta^3(\vec{r}) - \frac{qe^{-r/\lambda}}{4\pi r\lambda^2}}. \tag{3.3}$$

c) First by integrating the charge density over all of space.

$$Q = \int \rho\, dV = \int dV \left[q\delta^3(\vec{r}) - \frac{qe^{-r/\lambda}}{4\pi r\lambda^2}\right]$$
$$= q\left[1 - \int_0^\infty \frac{e^{-r/\lambda}}{4\pi r\lambda^2}4\pi r^2 dr\right]$$
$$= q\left[1 - \int_0^\infty xe^{-x}\,dx\right] = 0$$

Now by applying Gauss's law in integral form where r is the radius of a spherical Gaussian surface.

$$Q = \lim_{r\to\infty}\varepsilon_0 \oint \vec{E}\cdot d\vec{A}$$
$$= \lim_{r\to\infty}\varepsilon_0 \frac{q}{4\pi\varepsilon_0 r^2}\left[1+\frac{r}{\lambda}\right]e^{-r/\lambda}4\pi r^2$$

$$= \lim_{r\to\infty} q\left[1+\frac{r}{\lambda}\right]e^{-r/\lambda} = 0$$

4. Coaxial conductors

Calculate the energy per unit length for two very long coaxial cylindrical shells, neglecting edge effects. The inner and outer cylinders have radii a and b and uniform linear charge densities λ and $-\lambda$, respectively. The charge is uniformly distributed around the cylinder surfaces.

SOLUTION:
Here the cylindrical symmetry allows us to use cylindrical Gaussian surfaces to calculate the electric field in each of the regions $s < a$, $a < s < b$, and $s > b$. For both $s < a$ and $s > b$, there is no net charge enclosed in the Gaussian surface, so $\vec{E} = \vec{0}$ there. For $a < s < b$, we have

$$2\pi s l E = \frac{\lambda l}{\varepsilon_0} \Rightarrow \vec{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\,\hat{\mathbf{s}}$$

Now we can integrate the energy per unit length.

$$\frac{W}{l} = \frac{\varepsilon_0}{2l}\int E^2\,dV$$

$$= \frac{\varepsilon_0}{2l}\int_a^b \frac{\lambda^2}{4\pi^2\varepsilon_0^2 s^2}\cdot 2\pi l s\,ds$$

$$= \frac{\lambda^2}{4\pi\varepsilon_0}\int_a^b \frac{ds}{s}$$

$$\boxed{\frac{W}{l} = \frac{\lambda^2}{4\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)} \tag{4.1}$$

5. A spherical capacitor

A capacitor consists of two concentric spherical shells. The inner shell, conductor 1, has radius a, and the outer shell, conductor 2, has radius b. For this two-conductor system, find C_{11}, C_{12} and C_{22}.

SOLUTION: Case 1: set $V_2 = 0$ on the outer conductor while applying a voltage V_1 to the inner conducting sphere. Suppose we have charge q_2 on the outer conductor, and charge q_1 on the inner conductor. For the space between the two conductors, Gauss' law tells us that

$$\vec{E} = k\frac{q_1}{r^2}\hat{r},$$

so

$$V_2 - V_1 = -\int_a^b \vec{E}\cdot d\vec{\ell} = kq_1\left(\frac{1}{b}-\frac{1}{a}\right).$$

But here, $V_2 - V_1 = -V_1$, so

$$q_1 = 4\pi\varepsilon_0\frac{ab}{b-a}V_1$$

and hence

$$C_{11} = 4\pi\varepsilon_0\frac{ab}{b-a}.$$

The charge q_2 must cancel the potential contribution from q_1. Since for spherical geometries the electric field and potential both act like charges at the center of the sphere, we see we must have $q_2 = -q_1$ and therefore $C_{21} = -C_{11}$.

Case 2: set $V_1 = 0$ on the inner conductor while applying a voltage V_2 to the outer conducting sphere. From the same argument we started with above, we now have

$$V_2 - V_1 = V_2 = -\int_a^b \vec{E}\cdot d\vec{\ell} = kq_1\left(\frac{1}{b}-\frac{1}{a}\right),$$

and we see that $C_{12} = C_{21}$ as expected. For the outer sphere, we now have

$$V_2 = k\frac{q_1 + q_2}{b}.$$

Solving for q_2

$$q_2 = -\frac{b}{a}q_1.$$