Physics 3323, Fall 2016 — Problem Set 4 — *due Sep 23, 2016*

Reading: Griffiths 3.1 through 3.4

1. More spherical conducting shells

Two conducting spheres of radius a and b, each carrying a charge q, are separated by a distance $R \gg a, b$. Find the approximate potential of and the final charge on each sphere after they are connected by a fine conducting wire. Assume the following idealized conditions:

1. the charge distribution and the field of each sphere are radially symmetric,
2. each sphere can be regarded as a point charge relative to the location of the other sphere,
3. no charge resides on the wire.

(*Hints* – (i) the potential at the surface of each sphere may be regarded as the sum of two partial potentials: the potential due to the charge residing on the same sphere and the potential due the charge residing on the other sphere, and (ii) the total charge must be conserved.)

SOLUTION: Before the spheres are connected with the wire, they each have the same charge $q = q_a = q_b$, but in general they have different potentials. ($V_a \neq V_b$) After they are connected, charge will flow from one conductor to the other in order to make the potentials equal. ($V_a = V_b$) This charge will be conserved so that $q_a + q_b = 2q$.

Let's first write the potential at the spheres of radius a and b each as the sum of two terms. Let V_{ij} be the potential at sphere i due to sphere j.

$$\begin{aligned} V_a &= V_{aa} + V_{ab} \\ V_b &= V_{ba} + V_{bb} \end{aligned}$$

Now it is just a matter of writing the different V_{ij} in terms of the charges on the spheres and solving for the charges using the conditions $V_a = V_b$ and $q_a + q_b = 2q$.

$$V_{aa} = \frac{1}{4\pi\varepsilon_0}\frac{q_a}{a}, \quad V_{ab} = \frac{1}{4\pi\varepsilon_0}\frac{q_b}{R}$$

$$V_{ba} = \frac{1}{4\pi\varepsilon_0}\frac{q_a}{R}, \quad V_{bb} = \frac{1}{4\pi\varepsilon_0}\frac{q_b}{b}$$

The first condition leads to

$$\frac{1}{4\pi\varepsilon_0}\left(\frac{q_a}{a} + \frac{q_b}{R}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_a}{R} + \frac{q_b}{b}\right). \tag{1.1}$$

At this point we can use the fact that $R \gg a, b$ to ignore some terms and rewrite Eq. (**??**).

$$bq_a = aq_b \tag{1.2}$$

Now we use the second condition to write $q_b = 2q - q_a$.

$$bq_a = a(2q - q_a) \quad \Rightarrow \quad q_a = 2q\frac{a}{a+b}$$

q_b can then be found by symmetry.

$$\boxed{\begin{pmatrix} q_a \\ q_b \end{pmatrix} = \begin{pmatrix} 2q\frac{a}{a+b} \\ 2q\frac{b}{a+b} \end{pmatrix}} \tag{1.3}$$

2. A wire runs around it

We solved the problem of a point charge and a grounded metal plate conductor in lecture, and in so doing have in effect solved every problem that can be obtained from this by superposition! Suppose, for instance, that we have a straight wire 2000 m long that is uniformly charged with a linear density 10^{-4} C/m, and that runs parallel to the earth at a height of 5 m. For steady fields, the earth behaves like a good conductor. Neglect edge effects in the following.

a) What is the field strength at the surface of the earth, immediately below the wire?

b) What is the electric force acting on the wire?

c) If you follow a field line that starts out from the wire in a horizontal direction (that is, parallel to the earth), where does it meet the surface of the earth?

SOLUTION: This problem has a solution by the method of images, where the image is a straight wire 2000 m long that is uniformly charged with a linear density -10^{-4} C/m and runs parallel to the earth at a depth of 5 m below the surface.

a) We know a long line of charge has the field $\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{\mathbf{s}}$, where s is the distance from the line, and $\hat{\mathbf{s}}$ is a unit vector pointing away from the line. The contributions from the line of charge and the image charge at a point directly between them have the same magnitude and direction. Thus

$$\vec{E} = -2\frac{\lambda}{2\pi\varepsilon_0 h}\hat{\mathbf{y}}$$

$$\boxed{E = \frac{\lambda}{\pi\varepsilon_0 h} = \frac{10^{-4}\text{C/m}}{\pi\, 8.85\times 10^{-12}\text{F/m}\times 5\text{m}} = 719\,\text{kV/m}} \quad (2.1)$$

b)The force on the charged wire above the earth will be due to electric field produced by the induced charges in the earth plane. Since this electric field is the same as that from the image charge described earlier, we can just calculate the field due to the image charge at the position of the charge wire. The image charge has electric field

$$\vec{E} = -\frac{\lambda}{2\pi\varepsilon_0(2h)}\hat{\mathbf{y}}$$

at the position of the charged wire. This means that there is a force of $\lambda^2 dl/4\pi\varepsilon_0 h$ on a piece of charge λdl. This is true of dl anywhere except close to the ends of the wire. Since we are neglecting end effects, we can find the force by integrating over the line of charge.

$$F = \int \frac{\lambda^2\, dl}{4\pi\varepsilon_0 h} = \frac{\lambda^2 L}{4\pi\varepsilon_0 h}$$

The force is directed downward toward the ground.

$$\vec{F} = -\frac{\lambda^2 L}{4\pi\varepsilon_0 h}\hat{\mathbf{y}} = -\frac{(10^{-4}\text{ C/m})^2\times 2000\text{ m}}{4\pi\, 8.85\times 10^{-12}\text{ F/m}\times 5\text{ m}}\hat{\mathbf{y}}$$

$$\boxed{\vec{F} = -36.0\text{ kN }\hat{\mathbf{y}}} \quad (2.2)$$

c) Let the lines of charge run in the z-direction and let the y-direction be the vertical. I make the claim that the electric field lines are arcs of circles with centers on the x-axis which pass through the points $(0, h)$ and $(0, -h)$ where $h = 5$m. The electric field has the form

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{x\hat{\mathbf{x}} + (y-h)\hat{\mathbf{y}}}{x^2+(y-h)^2} - \frac{x\hat{\mathbf{x}} + (y+h)\hat{\mathbf{y}}}{x^2+(y+h)^2}\right]$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\left(\frac{x}{x^2+(y-h)^2}-\frac{x}{x^2+(y+h)^2}\right)\hat{\mathbf{x}}\right.$$
$$\left.+\left(\frac{y-h}{x^2+(y-h)^2}-\frac{y+h}{x^2+(y+h)^2}\right)\hat{\mathbf{y}}\right]$$
$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{2h\left[2xy\hat{\mathbf{x}}+\left(y^2-x^2-h^2\right)\hat{\mathbf{y}}\right]}{\left(x^2+(y-h)^2\right)\left(x^2+(y+h)^2\right)} \tag{2.3}$$

Thus we see that the vector field $\vec{E}$ is parallel to the vector field $2xy\hat{\mathbf{x}}+\left(y^2-x^2-h^2\right)\hat{\mathbf{y}}$. I will show that the unit vectors parallel to this vector field are the tangents to the family of circles centered on a point $(x_0, 0)$ on the x-axis passing through the points $(0, \pm h)$.
First, the unit vectors are

$$\hat{\mathbf{n}} = \frac{2xy\hat{\mathbf{x}}+\left(y^2-x^2-h^2\right)\hat{\mathbf{y}}}{\sqrt{\left(y^2-x^2-h^2\right)^2+(2xy)^2}}. \tag{2.4}$$

Now the family of curves can be described parametrically by

$$x = x_0 + r\cos\theta$$

$$y = r\sin\theta$$

where $r = \sqrt{x_0^2+h^2}$. These curves have tangent vectors

$$\hat{\mathbf{T}} = -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}.$$

By substituting the parametrization of the circles into Eq. (**??**), I will show that $\hat{\mathbf{n}} = \pm\hat{\mathbf{T}}$. The numerator of the x component.

$$2xy = 2r\sin\theta(x_0 + r\cos\theta)$$

The numerator of the y component.

$$y^2 - x^2 - h^2 = r^2\sin^2\theta - (x_0 + r\cos\theta)^2 - h^2$$
$$= r^2\sin^2\theta - r^2\cos^2\theta - 2rx_0\cos\theta - x_0^2 - h^2$$
$$= r^2\left(\sin^2\theta - \cos^2\theta - 1\right) - 2rx_0\cos\theta$$
$$= -2r^2\cos^2\theta - 2rx_0\cos\theta$$
$$= -2r\cos\theta\left(x_0 + r\cos\theta\right)$$

The square of the denominator.

$$\left(y^2-x^2-h^2\right)^2 + (2xy)^2 = 4r^2\cos^2\theta\left(x_0 + r\cos\theta\right)^2$$
$$+4r^2\sin^2\theta\left(x_0 + r\cos\theta\right)^2$$
$$= 4r^2\left(x_0 + r\cos\theta\right)^2 = \left(2r\left(x_0 + r\cos\theta\right)\right)^2$$

Thus we have for the unit vector $\hat{\mathbf{n}}$.

$$\hat{\mathbf{n}} = \frac{2xy\hat{\mathbf{x}}+\left(y^2-x^2-h^2\right)\hat{\mathbf{y}}}{\sqrt{\left(y^2-x^2-h^2\right)^2+(2xy)^2}}$$
$$= \frac{2r\left(x_0 + r\cos\theta\right)\left(\sin\theta\hat{\mathbf{x}} - \cos\theta\hat{\mathbf{y}}\right)}{\pm 2r\left(x_0 + r\cos\theta\right)}$$
$$= \pm\left(\sin\theta\hat{\mathbf{x}} - \cos\theta\hat{\mathbf{y}}\right) = \pm\hat{\mathbf{T}}$$

Now that we know that the field lines are arcs of these circles, it is clear that the circle of radius $h = 5$m is the one pertinent to the question. It passes through the x axis at a point 5m from directly below the axis.

3. Parallel plate capacitor, filled

A voltage V_0 is applied to a thin parallel plate capacitor of plate separation d filled with a cloud of positive charge density $\rho(x) = \rho_0\sin(\pi x/2d)$. The x-axis is perpendicular to the plates. Take the positive plate at $x = 0$ to be grounded and the negative plate at $x = d$ to have a potential $-V_0$.

a) Find the potential inside the capacitor with respect to the positive plate.

b) Find the electric field between the plates.

c) Find the surface charge density σ on the inner surfaces of the plates.

SOLUTION:

a) We need to solve the differential equation

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} = -\frac{\rho_0}{\varepsilon_0}\sin\left(\frac{\pi x}{2d}\right)$$

subject to the boundary conditions $V(x=0)=0$ and $V(x=d)=-V_0$. Since there is translational symmetry in the y and z directions, the problem becomes essentially a one-dimensional ODE in x.

$$\frac{d^2V}{dx^2} = -\frac{\rho_0}{\varepsilon_0}\sin\left(\frac{\pi x}{2d}\right) \tag{3.1}$$

A particular integral of Eq. (**??**) is

$$V_p(x) = V_1 \sin\left(\frac{\pi x}{2d}\right) \tag{3.2}$$

where

$$V_1 = \left(\frac{2d}{\pi}\right)^2 \frac{\rho_0}{\varepsilon_0}.$$

The homogeneous solution is

$$V_h(x) = C_1 x + C_2. \tag{3.3}$$

The full solution is then the sum of Eqs. (**??**–**??**).

$$V(x) = V_p + V_h = V_1 \sin\left(\frac{\pi x}{2d}\right) + C_1 x + C_2 \tag{3.4}$$

Now we use the boundary conditions to solve for C_1 and C_2.

$$V(0) = C_2 = 0$$

$$V(d) = V_1 + C_1 d = -V_0 \quad\Rightarrow\quad C_1 = -\frac{V_0+V_1}{d}$$

$$\boxed{V(x) = V_1\left[\sin\left(\frac{\pi x}{2d}\right) - \frac{x}{d}\right] - V_0\frac{x}{d}} \tag{3.5}$$

with V_1 defined as above.

b)

$$\vec{E} = -\vec{\nabla}V = \frac{dV}{dx}\hat{\mathbf{x}}$$

$$= \left(-V_1\left[\frac{\pi}{2d}\cos\left(\frac{\pi x}{2d}\right) - \frac{1}{d}\right] + \frac{V_0}{d}\right)\hat{\mathbf{x}}$$

$$\boxed{\vec{E} = \left(-\frac{2d\rho_0}{\pi\varepsilon_0}\cos\left(\frac{\pi x}{2d}\right) + \frac{V_1+V_0}{d}\right)\hat{\mathbf{x}}} \tag{3.6}$$

c) Here we use the electrostatic boundary condition

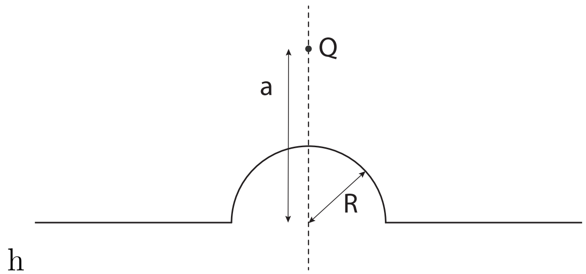
$$\frac{\partial V_{above}}{\partial n} - \frac{\partial V_b elow}{\partial n} = -\frac{1}{\varepsilon_0}\sigma.$$

We will take the n direction to be the x direction. Outside the conductors, the potential is constant (it satisfies Laplace's equation there). The surface charge at $x=0$ is

$$\boxed{\sigma_0 = -\varepsilon_0 \left.\frac{\partial V}{\partial x}\right|_0 = -\frac{2d\rho_0}{\pi} + \frac{\varepsilon_0}{d}(V_1+V_0)}$$

while the surface charge at $x=d$ is

$$\boxed{\sigma_d = \varepsilon_0 \left.\frac{\partial V}{\partial x}\right|_d = -\frac{\varepsilon_0}{d}(V_1+V_0).}$$



4. Imagine those charges

Griffiths Figure 3.12 shows a grounded metal sphere with a charge q outside of it. He argues that the method of images is applicable, you just have to put the correct charge (q') at the correct spot (point b inside the sphere).

a) Griffiths 3.8a (3.7a in 3rd ed.). This form makes explicit that your boundary conditions work.

b) Griffiths 3.8b (3.7b in 3rd ed.)

c) Now use this result to solve the more complex geometry shown above, with a hemispherical conducting bump (radius R) in an otherwise planar conducting sheet. A charge Q is sits a distance a above the plane of the sheet, centered above the bump. You can find the potential V anywhere in the plane above the conductor by using *three* image charges. What are their locations and charges? Be sure to explain your reasoning and to demonstrate that $V = 0$ on the entire conducting surface.

SOLUTION:

a) We want to show that

$$\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right) = \tag{4.1}$$

$$\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{q}{\sqrt{R^2 + \left(\frac{ra}{R}\right)^2 - 2ra\cos\theta}}\right]$$

The law of cosines can be used to show that the first terms on the LHS and RHS are equal. By the law of cosines,

$$\mathscr{r} = \sqrt{a^2 + r^2 - 2ar\cos\theta}$$

This shows that the first terms are equal. Now we need to show that the second term are also equal. First, we can rewrite $\mathscr{r}'$.

$$\mathscr{r}' = \sqrt{b^2 + r^2 - 2br\cos\theta}$$

Now use the relations $b = R^2/a$ and $q' = -qR/a$ to get

$$\begin{aligned}\frac{q'}{\mathscr{r}'} &= \frac{-q}{\frac{a}{R}\sqrt{\left(\frac{R^2}{a}\right)^2 + r^2 - 2\frac{R^2}{a}r\cos\theta}} \\ &= -\frac{q}{\sqrt{R^2 + \left(\frac{ra}{R}\right)^2 - 2ra\cos\theta}}.\end{aligned}$$

This shows that the second terms are also equal.

b) We will use the electrostatic boundary condition $\sigma = -\varepsilon_0\frac{\partial V}{\partial n}$ where the directional derivative is taken to be the outward normal of the conductor. In our case, $\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$.

$$\begin{aligned}\sigma &= -\frac{q}{4\pi}\frac{\partial}{\partial r}\left[\frac{1}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{1}{\sqrt{R^2 + \left(\frac{ra}{R}\right)^2 - 2ra\cos\theta}}\right]_{r=R} \\ &= -\frac{q}{4\pi}\left[-\frac{r - a\cos\theta}{(r^2 + a^2 - 2ra\cos\theta)^{3/2}} + \frac{ra^2/R^2 - a\cos\theta}{\left(R^2 + \left(\frac{ra}{R}\right)^2 - 2ra\cos\theta\right)^{3/2}}\right]_{r=R} \\ &= -\frac{q}{4\pi}\left[\frac{a\cos\theta - R}{(R^2 + a^2 - 2Ra\cos\theta)^{3/2}} + \frac{a^2/R - a\cos\theta}{(R^2 + a^2 - 2Ra\cos\theta)^{3/2}}\right]\end{aligned}$$

$$\boxed{\sigma = \frac{q}{4\pi}\left[\frac{R - a^2/R}{(R^2 + a^2 - 2Ra\cos\theta)^{3/2}}\right]} \tag{4.2}$$

Note that σ has the opposite sign of q since $R < a$. Since the sphere is grounded, there will be some net induced charge. We would expect the total induced charge to be equal to the value of the image charge. $Q = q' = -qR/a$ We want to calculate $\int \sigma\, dA$ over the surface of the sphere. In this case $dA = 2\pi R^2 \sin\theta d\theta$.

$$Q = \int_0^\pi \frac{q}{4\pi}\frac{R - a^2/R}{(R^2 + a^2 - 2Ra\cos\theta)^{3/2}} 2\pi R^2 \sin\theta\, d\theta$$

$$= \frac{qR^2}{2}\left(R - \frac{a^2}{R}\right)\int_0^\pi \frac{\sin\theta\, d\theta}{(R^2 + a^2 - 2Ra\cos\theta)^{3/2}}$$

Now make the substitution $x = R^2 + a^2 - 2Ra\cos\theta$, $dx = 2Ra\sin\theta d\theta$.

$$Q = \frac{q}{4a}\left(R^2 - a^2\right)\int_{(a-R)^2}^{(a+R)^2} x^{-3/2}\, dx$$

$$= \frac{q}{4a}\left(R^2 - a^2\right)(-2)\left[\frac{1}{a+R} - \frac{1}{a-R}\right]$$

$$\boxed{Q = \frac{q}{2a}\left(a^2 - R^2\right)\frac{a - R - a - R}{a^2 - R^2} = -q\frac{R}{a} = q'} \tag{4.3}$$

This is as we would expect.

c) If we put an image charge $q_1 = -QR/a$ at a point $z = b = R^2/a$ as before, then the potential will be 0 on the sphere of radius R. However, we still need to have a potential of 0 on the entire plane as well. This requires two more image charges, one each for the charge Q and q_1. Thus we have the following charges:

$$\begin{array}{ll} q_0 = Q: & \text{at } z = a \\ q_1 = -QR/a: & \text{at } z = b = R^2/a \\ q_2 = QR/a: & \text{at } z = -b = -R^2/a \\ q_3 = -Q: & \text{at } z = -a \end{array}$$

On the sphere, the contributions to the potential of q_0 and q_1 add to zero, while the contributions from q_2 and q_3 also add to zero. On the plane, the charges pair up differently. The contributions to the potential from q_0 and q_3 cancel, while those from q_1 and q_2 cancel. If you are still not satisfied with that, you can write the entire potential in a form similar to Eq. (**??**).

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{q}{\sqrt{R^2 + \left(\frac{ra}{R}\right)^2 - 2ra\cos\theta}} - \frac{q}{\sqrt{r^2 + a^2 + 2ra\cos\theta}} + \frac{q}{\sqrt{R^2 + \left(\frac{ra}{R}\right)^2 + 2ra\cos\theta}}\right] \tag{4.4}$$

From this expression it is clear that when $r = R$ or when $\theta = \pi/2$, $V = 0$.

5. A retake of Griffiths Example 3.4

Take the basic geometry of of Griffiths Example 3.4, but change it to a pipe of square cross section, with the origin located at the center of the pipe. Suppose the potential on this sides at $x = \pm a/2$ is 0, and the potential on the sides at $y = \pm a/2$ is $V_0\cos(\pi x/a)$. Find $V(x, y)$ inside the pipe, and make a rough sketch of the equipotential lines in the x, y plane.

SOLUTION: Here we have no charges in the region inside the pipe. There is translational symmetry in the z direction, so this is a two dimensional problem in x and y. We find the potential $V(x, y)$ by solving Laplace's equation with the appropriate boundary conditions.

$$\textbf{PDE:} \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{5.1}$$

$$V\left(\pm\frac{a}{2}, y\right) = 0$$

$$V\left(x, \pm\frac{a}{2}\right) = V_0 \cos\left(\frac{\pi x}{a}\right)$$

We will solve this problem by the method of separation of variables. We assume the separable solution $V(x, y) = X(x)Y(y)$. Inserting this into Eq. (**??**) yeilds.

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

Now we divide by $V(x, y) = X(x)Y(y)$.

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

The equation is now separated. The only way this can be true is if the left term and the right term are separately equal to constants which are additive inverses. Denoting the constant k^2 for reasons soon to be seen we get:

$$\frac{d^2X}{dx^2} = -k^2 X$$

$$\frac{d^2Y}{dy^2} = k^2 Y$$

These give the solutions

$$X(x) = A\cos\left(k\left(x - \frac{a}{2}\right)\right) + B\sin\left(k\left(x - \frac{a}{2}\right)\right)$$

and

$$Y(y) = C\cosh ky + D\sinh ky$$

Don't let the offset in the solutions for X throw you. They are perfectly good linearly independent solutions to the differential equation Eq. (**??**) but are in a form that will make applying the boundary conditions simpler. Speaking of boundary conditions, note that the entire situation is symetric in y. That is $V(x, -y) = V(x, y)$. This implies that $D = 0$. Our solutions now read:

$$X(x) = A\cos\left(k\left(x - \frac{a}{2}\right)\right) + B\sin\left(k\left(x - \frac{a}{2}\right)\right)$$

$$Y(y) = C\cosh ky$$

Applying the boundary condition at $x = a/2$ gives

$$X(a/2) = 0 = A$$

while applying the boundary condition at $x = -a/2$ gives

$$X(-a/2) = 0 = -B\sin ka.$$

Now we could let $B = 0$, but that would give 0 for the entire solution. Thus $k = n\pi/a$ where n is an integer. In order to apply the final boundary condition, we need to take a linear combination of the valid solutions, one for each n.

$$V(x, y) = \sum_n X_n(x)Y_n(y) = \sum_n B_n \cosh\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a} - \frac{n\pi}{2}\right) \tag{5.2}$$

$$V\left(x, \pm\frac{a}{2}\right) = V_0\cos\left(\frac{\pi x}{a}\right)$$

$$= \sum_n B_n \cosh\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{a} - \frac{n\pi}{2}\right)$$

Now we could use Fourier's trick to integrate over othogonal functions here, but it is a little simpler to notice that for $n = 1$

$$\sin\left(\frac{\pi x}{a} - \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) - \frac{\pi x}{a} = -\cos\left(\frac{\pi x}{a}\right).$$

If we set $B_n = 0$ for all $n \neq 1$, and let $B_1 = -V_0/\cosh(\pi/2)$ we get

$$V\left(x, \pm\frac{a}{2}\right) = -V_0 \frac{\cosh(\pi/2)}{\cosh(\pi/2)} \sin\left(\frac{\pi x}{a} - \frac{\pi}{2}\right)$$

$$= -V_0 \sin\left(\frac{\pi x}{a} - \frac{\pi}{2}\right) = V_0 \cos\left(\frac{\pi x}{a}\right)$$

as required. This makes the full solution

$$\boxed{V(x, y) = V_0 \frac{\cosh(\pi y/a)}{\cosh(\pi/2)} \cos\left(\frac{\pi x}{a}\right)} \tag{5.3}$$