Physics 3323, Fall 2016 Problem Set 5 *due September 30, 2016*

Reading: Griffiths 4.1 through 4.4.3

1. Capacitance

In class, we investigated the potential for a conducting sphere whose top half is held at V_0 and whose bottom half (electrically isolated from the top) is held at $-V_0$. We found

$$V(r<R,\theta) = V_0 \sum_{\ell\,\mathrm{odd}} (2\ell+1)\alpha_\ell \left(\frac{r}{R}\right)^{\ell} P_\ell(\cos\theta)$$

$$V(r>R,\theta) = V_0 \sum_{\ell\,\mathrm{odd}} (2\ell+1)\alpha_\ell \left(\frac{R}{r}\right)^{\ell+1} P_\ell(\cos\theta)$$

where

$$\alpha_l = \int_0^1 P_\ell(x)\,dx = \frac{1}{\ell(\ell+1)} \left.\frac{dP_\ell(x)}{dx}\right|_{x=0}.$$

Find the capacitance of this device.

SOLUTION: The capacitance $C = Q/V$. The charge density is given by

$$\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)_{r=R} = -\frac{\sigma}{\epsilon_0}$$

Using the equations above we write

$$\begin{aligned}
-\frac{\sigma(\theta)}{\epsilon_0} &= V_0 \sum_{l\ odd} (2l+1)\alpha_l P_l(\cos\theta)\left(-(l+1)\frac{R^{l+1}}{r^{l+2}} - l\frac{r^{l-1}}{R^l}\right)_{r=R} \\
&= V_0 \sum_{l\ odd} (2l+1)\alpha_l P_l(\cos\theta)\left(-(l+1)\frac{R^{l+1}}{R^{l+2}} - l\frac{R^{l-1}}{R^l}\right) \\
&= V_0 \sum_{l\ odd} (2l+1)\alpha_l P_l(\cos\theta)\left(\frac{-(2l+1)}{R}\right)
\end{aligned}$$

The total charge on the upper half sphere is

$$\begin{aligned}
2\pi R^2 \int_0^{\pi/2} \sigma(\theta)\sin\theta d\theta &= \epsilon_0 V_0 \frac{2\pi R^2}{R} \\
&\quad \times \sum_{l\ odd} (2l+1)^2 \alpha_l \int_0^{\pi/2} P_l(\cos\theta)\sin\theta d\theta \\
&= \epsilon_0 V_0 \frac{2\pi R^2}{R} \times \sum_{l\ odd} (2l+1)^2 \alpha_l \int_0^1 P_l(x)dx \\
Q &= \epsilon_0 V_0 (2\pi R) \times \sum_{l\ odd} (2l+1)^2 \alpha_l^2
\end{aligned}$$

Then

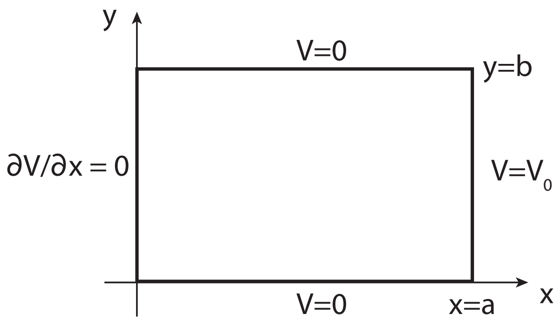
$$C = \frac{Q}{\Delta V} = \frac{Q}{2V_0} = \epsilon_0(\pi R) \times \sum_{l\ odd} (2l+1)^2 \alpha_l^2$$

2. A derivative boundary condition

You have a potential, invariant in z, for which you know the boundary conditions as shown in the rectangle in the figure below. (*From Heald and Marion*)

a) Find the potential for all points inside the rectangle.

b) Sketch a few field-lines and equipotentials.



SOLUTION:

a)The general solution is

$$V(x,y) = X(x)Y(y) = (Ae^{kx} + Be^{-kx})(C\cos ky + D\sin ky)$$

The boundary conditions at $y = 0$ and $y = b$ are satisfied if $k = \frac{n\pi}{b}$ and $C = 0$. The boundary condition at $x = 0$ $\frac{\partial V}{\partial x} = 0$ implies that

$$\left(kAe^{kx} - kBe^{-kx}\right)_{x=0} = 0 \rightarrow A = B.$$

Therefore

$$V(x,y) = \sum_{n=1}^{\infty} A_n \cosh(\frac{n\pi}{b}x)\sin\frac{n\pi}{b}y$$

The boundary condition at $x = a$ requires that

$$V_0 = \sum_{n=1}^{\infty} A_n \cosh(\frac{n\pi}{b}a)\sin\frac{n\pi}{b}y$$

$$\int_0^b V_0 \sin(\frac{n'\pi}{b}y)dy = \sum_{n=1}^{\infty} A_n \cosh\frac{n\pi}{b}a \int_0^b \sin(\frac{n'\pi}{b}y)\sin(\frac{n\pi}{b}y)dy$$

$$2V_0\frac{b}{n'\pi} = \sum_{n=1}^{\infty} A_n \cosh(\frac{n\pi}{b}a)\frac{b}{2}\delta_{nn'}$$

$$\rightarrow A_n = \frac{4V_0}{n\pi\cosh(\frac{n\pi a}{b})}$$

$$V = \sum_{n=1}^{\infty} \frac{4V_0\cosh(\frac{n\pi x}{b})}{n\pi\cosh(\frac{n\pi a}{b})}\sin(\frac{n\pi}{b}y)$$

b)See Figure 1.

3. Charged Conducting Sphere

Suppose you have a uniform electric field $\vec{E} = E_0\hat{z}$ into which you will place a spherical conductor of radius R, as in Griffiths Example 3.8. The conducting sphere in this problem carries a total charge Q.

a) Find the potential in the region outside the sphere.

b) Determine the charge density on the sphere, and interpret what you find.

SOLUTION: The potential inside the sphere is the same as the potential on the sphere since it is a conductor. The

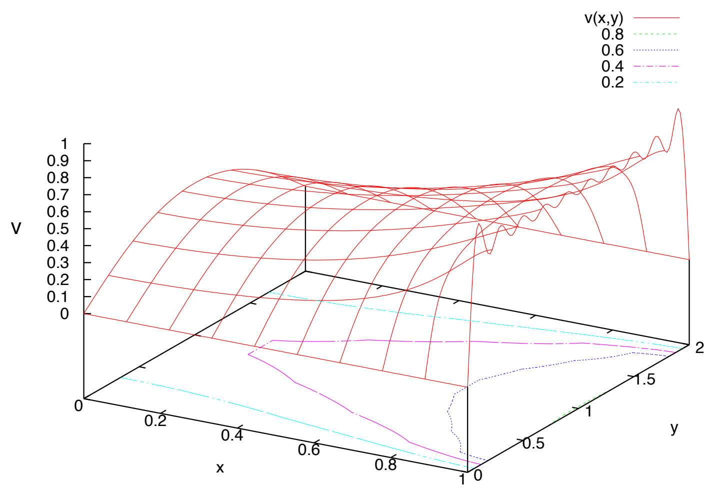


Figure 1: $a = 1$,$b = 2$,$V_0 = \pi/4$, sum includes all odd n from 1,19

general solution for the potential outside the sphere is

$$V_{out} = \sum_{l=0}^{\infty}(A_l r^l + \frac{B_l}{r^{l+1}})P_l(\cos\theta)$$

The potential of the sphere is a constant, V_0.

$$\begin{aligned} V_0 &= \sum_{l=0}^{\infty}(A_l R^l + \frac{B_l}{R^{l+1}})P_l(\cos\theta) \\ \int_0^{\pi} V_0 P_{l'}(\cos\theta)\sin\theta d\theta &= \sum_{l=0}^{\infty}(A_l R^l + \frac{B_l}{R^{l+1}})\int_0^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)d\theta \end{aligned}$$

The integral on the left is 0, unless $l' = 0$. Therefore each term in the sum is 0 except $l' = 0$ and when $l' = 0$ we get

$$\begin{aligned} 2V_0 &= \sum_{l=1}^{\infty}(A_l R^l + \frac{B_l}{R^{l+1}})\frac{2\delta_{l0}}{2l+1} \\ V_0 &= (A_0 + \frac{B_0}{R}) \end{aligned}$$

Meanwhile for all the other terms we have

$$A_l R^l + \frac{B_l}{R^{l+1}} = 0 \rightarrow B_l = -A_l R^{2l+1}$$

At $r \rightarrow \infty, V \rightarrow -E_0 z = -E_0 r\cos\theta$ so that $E = -\nabla V = E_0\hat{\mathbf{z}}$. Therefore $A_1 P_1(\cos\theta) = -E_0\cos\theta \rightarrow A_1 = -E_0$ and $A_{l>1} = 0$, $B_1 = R^3 E_0$ and $A_0 = 0$ and $B_0 = RV_0$. Finally

$$V = (A_1 r + \frac{B_1}{r^2})\cos\theta + \frac{B_0}{r} = -E_0(r - \frac{R^3}{r^2})\cos\theta + V_0\frac{R}{r}$$

The surface charge density on the sphere is

$$\frac{\sigma(\theta)}{\epsilon_0} = -\frac{\partial V_{out}}{\partial r} = E_0(1 + 2\frac{R^3}{R^3})\cos\theta + V_0\frac{R}{R^2}$$

The total charge

$$Q = 2\pi R^2\epsilon_0\int_0^{\pi}(3E_0\cos\theta + V_0\frac{R}{R^2})\sin\theta d\theta = 4\pi\epsilon_0 R V_0$$

and

$$V_0 = \frac{Q}{4\pi\epsilon_0 R}$$

4. A spherical insulator

A very thin insulating spherical shell of radius R_0 has an axially symmetric surface charge density $\sigma(\theta) = \sigma_0(3\cos^2\theta - 1)$

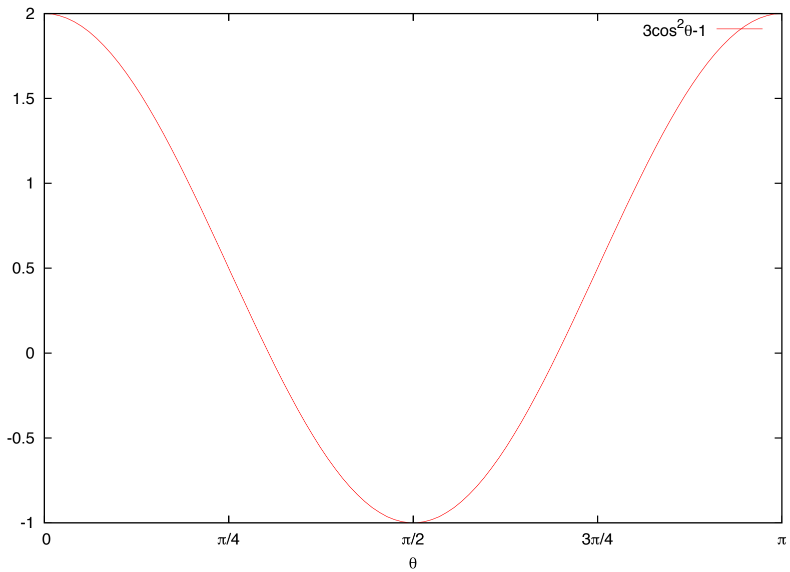
distributed over its surface. The polar angle θ is measured with respect to the z axis and σ_0 is a constant.

a) **S**ketch the charge distribution as a function of θ.

b) **F**ind the potential both inside and outside the shell.

SOLUTION:

a)



b)The potentials inside and outside the sphere are

$$V_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$
$$V_{out} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

The potential is continuous at the boundary. Therefore

$$V_{in}(R) = V_{out}(R) \rightarrow A_l R^l = \frac{B_l}{R^{l+1}}$$

The charge density

$$\sigma = -\epsilon_0 \left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r} \right)_{r=R}$$
$$2\sigma_0 P_2(\cos\theta) = -\epsilon_0 \sum_{l=0}^{\infty} P_l(\cos\theta) \left(-(l+1)\frac{B_l}{R^{l+2}} - l A_l R^{l-1} \right)$$

where we note that $\sigma = \frac{\sigma_0}{2} P_2(\cos\theta)$. Then substituting $B_l = A_l R^{2l+1}$ we get

$$2\sigma_0 P_2(\cos\theta) = -\epsilon_0 \sum_{l=0}^{\infty} P_l(\cos\theta) \left(-(l+1)\frac{A_l R^{2l+1}}{R^{l+2}} - l A_l R^{l-1} \right)$$
$$= -\epsilon_0 \sum_{l=0}^{\infty} P_l(\cos\theta) \left(-(2l+1) A_l R^{l-1} \right)$$

It is clear that only A_2 is nonzero. Then

$$A_2 = \frac{2\sigma_0}{5\epsilon_0 R}$$

and

$$V_{in} = \frac{2\sigma_0}{5\epsilon_0 R} r^2 P_2(\cos\theta)$$

$$V_{out} = \frac{B_l}{r^2}P_2 = \frac{2\sigma_0}{5\epsilon_0 R}\frac{R^5}{r^3}P_2(\cos\theta) = \frac{2\sigma_0 R^4}{5\epsilon_0 r^3}P_2(\cos\theta)$$

5. Axial expansion and a ring of charge

Starting from our results for separation of variables in spherical coordinates for an axially symmetric distribution, we can obtain the axial expansion, a method of determining the potential everywhere for symmetric charge distribution when we know the potential on the axis of symmetry. Consider our general result

$$V(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + B_n/r^{n+1}\right)P_n(\cos\theta), \qquad (5.1)$$

where the z axis is the symmetry axis. On the symmetry axis, for which $\theta = 0$, $P_n(1) = 1$ for all n. Suppose that the potential on the axis is known, and can be expressed as a power series in z:

$$V_{\text{axis}}(z) = \sum_{n=0}^{\infty}\left(a_n z^n + b_n/z^{n+1}\right). \qquad (5.2)$$

This is exactly the same form as our general result evaluated on the z axis, so the coefficients in the two series must be equal. If we know the expansion on the z axis, it immediately tells us the full expansion!

a) Using problem 1 in Problem Set 01 as your starting point, find the potential of a thin, uniformly charged, circular ring of radius a and total charge q at all points for which $r > a$, where r is the distance from the center of the ring. In particular, evaluate the first three terms of the expansion.

b) Sketch your result.

c) Comparing your result to the multipole expansion in terms of Legendre polynomials in Griffiths, identify the multipoles to which each of your three terms belong.

SOLUTION:

a)The potential of a thin uniformly charged circular ring of radius a and total charge q on the z-axis is

$$\begin{aligned} V(r) &= \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{r^2+a^2}} \\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{r\sqrt{1+a^2/r^2}} \\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{r}\left(1 - \frac{1}{2}\frac{a^2}{r^2} + \frac{3}{8}\left(\frac{a^2}{r^2}\right)^2 - \frac{5}{16}\left(\frac{a^2}{r^2}\right)^3 + \ldots\right) \\ \rightarrow & \\ V(r,\theta) &= \frac{q}{4\pi\epsilon_0 r}\left(1 - \frac{1}{2}\frac{a^2}{r^2}P_2(\cos\theta) + \frac{3}{8}\frac{a^4}{r^4}P_4(\cos\theta) + \frac{5}{16}\frac{a^6}{r^6}P_6(\cos\theta) + \ldots\right. \end{aligned}$$

b)See Figure 2.

6. Leading behavior of various line charges

Griffiths 3.46 (3.40 in 3rd ed.)

SOLUTION:

a)$\lambda = k\cos(\frac{\pi z}{2a})$. The monopole term is

$$\int_{-a}^{a} k\cos(\frac{\pi z}{2a})dz = \frac{2ak}{\pi}\sin(\frac{\pi z}{2a})|_{-a}^{a} = \frac{4ak}{\pi}$$

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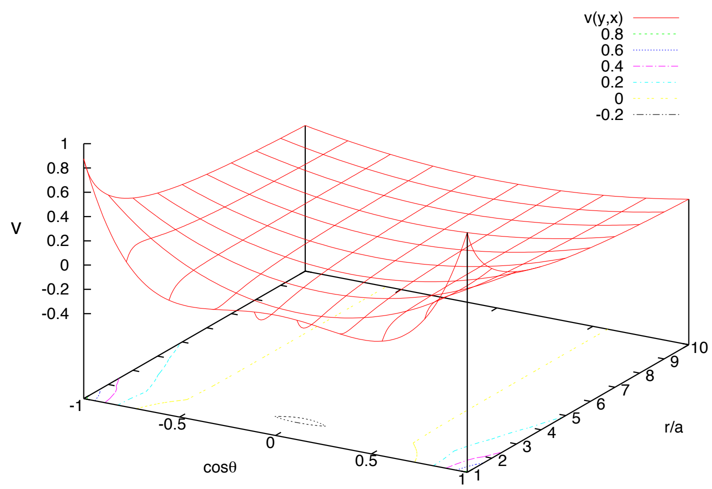

Figure 2: First 3 terms in the expansion

b)$\lambda = k\sin(\pi z/a)$. The monopole term is $\int_{-a}^{a} k\sin(\frac{\pi z}{a})dz = 0$. The dipole term is

$$\int_{-a}^{a} k\sin(\frac{\pi z}{a})zdz = \frac{k}{\pi^2}\left(a^2\sin(\pi z/a) - \pi a z\cos(\pi z/a)\right)_{-a}^{a}$$

$$= 2ka^2/\pi$$

c)$\lambda = k\cos(\frac{\pi z}{a})$. The monopole term is $\int_{-a}^{a} k\cos(\frac{\pi z}{a})dz = 0$. The dipole term is

$$\int_{-a}^{a} k\cos(\frac{\pi z}{a})zdz = 0$$

. The quadrupole term

$$\int_{-a}^{a} k\cos(\frac{\pi z}{a})z^2dz = k\frac{4a^3}{\pi^2}$$

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