Physics 3323, Fall 2016

Problem Set 7

Reading: Griffiths 4.1 through 4.4.1

1. Electric dipole

An electric dipole with $\vec{p} = -p_0 \hat{z}$ is located at the origin and is sitting in an otherwise uniform electric field $\vec{E} = E_0 \hat{z}$. Find the spherical surface, centered at the origin, through which no electric field lines pass, that is for which $\vec{E} \cdot \hat{n} = 0$. What is its radius?

SOLUTION: The electric field of a dipole is given by Griffiths in equation 3.103. We will use this form except in our case, the dipole is $\vec{p} = -p_0 \hat{z}$ so that we get

$$\vec{E}_{dip} = -\frac{p_0}{4\pi\varepsilon_0 r^3} \left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right). \tag{1.1}$$

For the uniform field, we just need to replace $\hat{\mathbf{z}}$ with $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\theta}$. The total field becomes

$$\vec{E} = \left(E_0 - \frac{2p_0}{4\pi\varepsilon_0 r^3}\right)\cos\theta\hat{\mathbf{r}} - \left(E_0 + \frac{p_0}{4\pi\varepsilon_0 r^3}\right)\sin\theta\hat{\theta}.$$
 (1.2)

We want to find the spherical surface where $\vec{E} \cdot \hat{\mathbf{n}} = \mathbf{0}$.

$$\vec{E} \cdot \hat{\mathbf{n}} = \tilde{\mathbf{E}} \cdot \hat{\mathbf{r}} = \left(\mathbf{E_0} - \frac{2\mathbf{p_0}}{4\pi\varepsilon_0 \mathbf{r^3}} \right) \cos\theta = \mathbf{0} \qquad (1.3)$$

$$r = \left(\frac{p_0}{2\pi\varepsilon_0 E_0}\right)^{1/3} \tag{1.4}$$





Figure 1: Diagram of dipole and ion for problem 2.

2. Attractive ions

Between any ion and a neutral atom, there is a force that arises in the following manner. The electric field of the ion polarizes the atom, and the field of that induced dipole reacts on the ion.

a) Show that this force is always attractive, and that it varies with the inverse *fifth* power of the distance of separation r.

b) Derive an expression for the associated potential energy, with zero energy corresponding to infinite separation. **c)** For what distance r is this potential energy of the same magnitude as kT at room temperature if the ion is singly charged (q = e) and the atom is a sodium atom?

SOLUTION:

a) Let us place the polarized atom at the origin, and arrange our coordinate axes so that the ion is on the z-axis a distance r away. We want to know the force the polarized atom exerts on the ion, so first we need to know how polarized the atom is. This will depend on the electric field at the origin from the ion, so we start by calculating the ion's electric field at the origin.

$$\vec{E}_{ion}(0) = -\frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{\mathbf{z}}$$
(2.1)

Now the induced dipole moment of the atom is given by $\vec{p} = \alpha \vec{E}$. Now we can calculate the polarization of the atom.

$$\vec{p}_{atom} = \alpha \vec{E}_{ion} = -\frac{1}{4\pi\varepsilon_0} \frac{\alpha q}{r^2} \hat{\mathbf{z}}$$
(2.2)

We have the dipole moment of the atom, and our system is set up in such a way that it points in the (negative) zdirection. We can use equation 3.103 from Griffiths with $p = -\alpha q/4\pi\varepsilon_0 r^2$ and $\theta = 0$ to write down the electric field from the dipole at the position of the ion.

$$\vec{E}_{dip}(r\hat{\mathbf{z}}) = \frac{\mathbf{p}}{4\pi\varepsilon_0 \mathbf{r}^3} \left(2\cos\theta \hat{\mathbf{r}} + \sin\theta \hat{\theta} \right) = \frac{2\mathbf{p}}{4\pi\varepsilon_0 \mathbf{r}^3} \hat{\mathbf{z}} \quad (2.3)$$

$$\vec{E}_{dip}(r\hat{\mathbf{z}}) = -\frac{2\alpha \mathbf{q}}{(4\pi\varepsilon_0)^2 \mathbf{r}^5}\hat{\mathbf{z}}$$
(2.4)

The force on the ion is then given by $\vec{F} = q\vec{E}$.

$$\vec{F} = -\frac{2\alpha q^2}{(4\pi\varepsilon_0)^2 r^5} \hat{\mathbf{z}}$$
(2.5)

We see that the force on the ion is directed towards the atom, i.e. it is attractive, and that it varies with the inverse fifth power of r.

b) We can get the solution for the potential energy by integrating the force along a path starting at infinity.

$$U = -\int_{\infty}^{r} F_r dr = \int_{\infty}^{r} \frac{2\alpha q^2}{(4\pi\varepsilon_0)^2 r^5} dr \qquad (2.6)$$
$$U = -\frac{\alpha q^2}{2(4\pi\varepsilon_0)^2 r^4} \qquad (2.7)$$

3. Get a charge out of polarization Griffiths 4.10

SOLUTION:
a)

$$\sigma_{b} = \vec{P} \cdot \hat{\mathbf{n}} = \tilde{\mathbf{P}} \cdot \hat{\mathbf{r}} \Big|_{\mathbf{r}=\mathbf{R}} = \mathbf{k}\mathbf{r}\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} \Big|_{\mathbf{r}=\mathbf{R}}$$
(3.1)

$$\rho_{b} = -\nabla \cdot \vec{P} = -\frac{1}{r^{2}} \frac{\partial}{\partial r} [r^{2}P_{r}] = -\frac{1}{r^{2}} \frac{\partial}{\partial r} [kr^{3}] = -\frac{k}{r^{2}} 3r^{2}$$
(3.2)

$$\boxed{\sigma_{b} = kR}}_{\rho_{b} = -3k}$$
(3.3)

b) We have spherical symmetry, so we can use Gauss's Law in integral form. Since there is no free charge, the total charge is just the bound charge. Because of the sperical symmetry, $\vec{E} = E\hat{\mathbf{r}}$.

Inside the Sphere:

$$4\pi r^2 E_{in} = \frac{4}{3}\pi r^3 \frac{\rho_b}{\varepsilon_0} \implies E_{in} = \frac{4\pi r^3/3}{4\pi \varepsilon_0 r^2} (-3k) = -\frac{kr}{\varepsilon_0} \quad (3.4)$$

Outside the Sphere:

$$4\pi r^2 E_{out} = \frac{4}{3}\pi R^3 \frac{\rho_b}{\varepsilon_0} + 4\pi R^2 \frac{\sigma_b}{\varepsilon_0} = -\frac{4\pi R^3}{\varepsilon_0} + \frac{4\pi R^3}{\varepsilon_0} = 0 \quad (3.5)$$

$$\begin{vmatrix} \vec{E}_{in} = -\frac{kr}{\varepsilon_0} \\ \vec{E}_{out} = 0 \end{vmatrix}$$
(3.6)

4. Permanently polarized dielectric shell

Griffiths 4.15

SOLUTION:

a) There are two surfaces which for which we need to calculate the bound surface charge. We also need to calculate the bound charge density in between the two.

Inner Surface:

$$\sigma_{b,a} = \vec{P} \cdot \hat{\mathbf{n}} = -\tilde{\mathbf{P}} \cdot \hat{\mathbf{r}} = -\frac{\mathbf{k}}{\mathbf{a}}$$
(4.1)

Outer Surface:

$$\sigma_{b,b} = \vec{P} \cdot \hat{\mathbf{r}} = \frac{k}{b} \tag{4.2}$$

In between:

$$\rho_b = -\nabla \cdot \vec{P} = -\frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{k}{r} \right] = -\frac{k}{r^2} \tag{4.3}$$

We have spherical symmetry so $\vec{E} = E\hat{\mathbf{r}}$. We can use Gauss's law in integral form to calculate E.

For r < a:

$$4\pi r^2 E = 0 \quad \Rightarrow \vec{E} = 0 \tag{4.4}$$

For a < r < b:

$$4\pi r^2 E = \frac{4\pi a^2(-k/a)}{\varepsilon_0} + \frac{1}{\varepsilon_0} \int_a^r \left(-\frac{k}{r^2}\right) 4\pi r^2 \, dr \qquad (4.5)$$

$$E = \frac{1}{r^2} \left[-\frac{ka}{\varepsilon_0} - \frac{k(r-a)}{\varepsilon_0} \right] = -\frac{k}{r\varepsilon_0}$$
(4.6)

For r > b:

$$4\pi r^{2}E = \frac{4\pi a^{2}(-k/a)}{\varepsilon_{0}} + \frac{1}{\varepsilon_{0}} \int_{a}^{b} \left(-\frac{k}{r^{2}}\right) 4\pi r^{2} dr + \frac{4\pi b^{2}(k/b)}{\varepsilon_{0}}$$
(4.7)
$$E = \frac{1}{r^{2}} \left[-\frac{ka}{\varepsilon_{0}} - \frac{k(b-a)}{\varepsilon_{0}} = -\frac{k}{r\varepsilon_{0}} + \frac{kb}{\varepsilon_{0}}\right] = 0$$
(4.8)

All together we have the following solution:

$$\begin{vmatrix} \sigma_{b,a} &= -k/a \\ \sigma_{b,b} &= k/b \\ \vec{E} &= \begin{cases} -\frac{k}{r\varepsilon_0} \hat{\mathbf{r}} & \text{for } a < r < b \\ 0 & \text{otherwise} \end{cases}$$
(4.9)

b) Since there is no free charge but we still have spherical symmetry, we find from Gauss's law that $\vec{D} = 0$ everywhere.

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = 0 \implies \vec{E} = -\frac{1}{\varepsilon_0} \vec{P} \qquad (4.10)$$

$$\vec{E} = \begin{cases} -\frac{k}{r\varepsilon_0} \hat{\mathbf{r}} & \text{for } a < r < b\\ 0 & \text{otherwise} \end{cases}$$
(4.11)

5. Hemispherical capacitor

A hemispherical conducting shell (radius b) is filled with a soft plastic, characterized by a very large relative dielectric constant ε_r and a very small electrical conductivity σ . A needleshaped conductor with a hemispherical tip (radius a) is pressed into the plastic, as shown in the figure, so that it is concentric with the shell. When the circuit switch is closed, a very small



Figure 2: Capacitor for problem 5.

current I flows and charges +Q and -Q appear on the conductors.

a) What is the capacitance of this arrangement?

b) Find the magnitude $\left| \vec{E} \right|$ of the electric field in the volume between the two conductors as a function of the distance r from the center of the conductors.

c) FInd the surface density of bound charge on the inner

(r = a) and the outer (r = b surfaces of the dielectric.

d) What is the surface density of bound charge on the flat surface of the dielectric? Explain.

SOLUTION:

a) Since a full spherical capacitor is just two hemispherical capacitors in parallel, a hemispherical capacitor has half the capacitance of a spherical capacitor. We can calculate the

capacitance of two concentric spheres filled with a dielectric.

$$\vec{D} = \frac{Q}{4\pi r^2} \hat{\mathbf{r}} \Rightarrow \vec{E} = \frac{Q}{4\pi \varepsilon_r \varepsilon_0 r^2} \hat{\mathbf{r}}$$
(5.1)

$$\Delta V = \int_{a}^{b} E_{r} dr = \frac{Q}{4\pi\varepsilon_{r}\varepsilon_{0}} \int_{a}^{b} \frac{dr}{r^{2}} = \frac{Q}{4\pi\varepsilon_{r}\varepsilon_{0}} \left(\frac{1}{a} - \frac{1}{b}\right) \quad (5.2)$$
$$C = \frac{Q}{\Delta V} = 4\pi\varepsilon_{r}\varepsilon_{0}\frac{ab}{b-a} \quad (5.3)$$

For the hemisphere:

$$C = 2\pi\varepsilon_r\varepsilon_0 \frac{ab}{b-a}$$
(5.4)

Because the dielectric has a large dielectric constant, we can ignore the edge effects due to the hemisphere ending and breaking spherical symmetry.

b) The electric field was found in part a) in Eq. (??).

$$E = \frac{Q}{4\pi\varepsilon_r\varepsilon_0 r^2} \tag{5.5}$$

c)

$$\sigma_b = \vec{P} \cdot \hat{\mathbf{n}} = (\tilde{\mathbf{D}} - \varepsilon_0 \tilde{\mathbf{E}}) \cdot \hat{\mathbf{n}}$$
(5.6)

$$\sigma_{b,a} = -\vec{P} \cdot \hat{\mathbf{r}} = -\frac{Q}{4\pi a^2} \left(1 - \frac{1}{\varepsilon_r}\right)$$
(5.7)

$$\sigma_{b,b} = \vec{P} \cdot \hat{\mathbf{r}} = \frac{Q}{4\pi b^2} \left(1 - \frac{1}{\varepsilon_r} \right)$$
(5.8)

d) There will be no surface bound charge on the flat part of the dielectric since \vec{E} , \vec{D} , and \vec{P} all point radially.

6. Cylindrical capacitor in dielectric oil

Griffiths 4.28

SOLUTION: Griffiths 4.64 on page 195 gives a general expression for the force exerted on a moveable dielectric.

$$F = \frac{1}{2}V^2 \frac{dC}{dx} \tag{6.1}$$

We can calculate the force per unit area of the oil in the capacitor, and equate it to the pressure in a fluid at depth x = h to find h.

$$P = \rho g h \tag{6.2}$$

First though, we need to find the capacitance of the capacitor so we can take its derivative. Suppose that before the oil rises there is a (long) length L_{oil} under the oil and a (similarly long) length L_{air} above the oil. What is the capacitance of the cylindrical capacitor? We can find the electric field inside the capacitor as a function of Q, the charge on the inner cylinder, using Gaus's law and then integrate it to get the potential difference between the two cylinders.

$$2\pi r L E = \frac{Q}{\varepsilon_0} \Rightarrow \vec{E} = \frac{Q/L}{2\pi s \varepsilon_0} \hat{\mathbf{s}}$$
(6.3)

$$\Delta V = -\int_{b}^{a} \frac{Q/L}{2\pi s\varepsilon_{0}} \hat{\mathbf{s}} \cdot ds \hat{\mathbf{s}} = \frac{Q/L}{2\pi\varepsilon_{0}} \ln\left(\frac{b}{a}\right) \qquad (6.4)$$

Thus we find that the capacitance of the part of the capacitor that is in air.

$$C_{air} = \frac{Q_{air}}{\Delta V} = \frac{2\pi\varepsilon_0}{\ln b/a} \left(L_{air} - x \right) \tag{6.5}$$

Here $Q_{air} = Q (L_{air} - x) / L$. The part of the capacitor in the oil with suceptibility χ_e has a different capacitance.

$$\frac{C}{L} = (1 + \chi_e) \frac{2\pi\varepsilon_0}{\ln b/a} \left(L_{oil} + x \right) \tag{6.6}$$

Now we can calculate $\frac{dC}{dx}$.

$$\frac{dC}{dx} = \frac{2\pi\varepsilon_0}{\ln b/a} \left(-1 + 1 + \chi_e\right) = \frac{2\pi\varepsilon_0\chi_e}{\ln b/a} \tag{6.7}$$

$$F = \frac{1}{2}V^2 \frac{dC}{dx} = \frac{\pi\varepsilon_0 \chi_e V^2}{\ln b/a}$$
(6.8)

Now we equate the pressure to the force per unit area. $A = \pi (b^2 - a^2).$

$$P = \rho g h = \frac{F}{A} = \frac{\pi \varepsilon_0 \chi_e V^2}{\pi (b^2 - a^2) \ln b/a}$$
(6.9)

$$h = \frac{\varepsilon_0 \chi_e V^2}{\rho g (b^2 - a^2) \ln b/a}$$
(6.10)

7. Current Loop

A circular current loop with radius *a* lies in the *xy* plane, carries a current *I*, and is centered at the origin. Application of the Biot-Savart law in Griffiths Example 5.6 gives the magnetic field of the current loop along the *z*-axis in the form $B_z(z)\hat{z}$. **a)** Evaluate $\int_{-\infty}^{+\infty} B_z dz$.

b) Now use Ampère's law in the form $\oint \vec{B} \cdot \vec{\ell}$ for the closed path that extends along the z axis from $z = -\infty$ to $z = +\infty$, and then returns along a semicircular arc "at" $r = +\infty$. What are the ingredients that go into the integral $\int_{\text{arc}} \vec{B} \cdot \vec{\ell}$ in the limit that $r \gg a$. Find the limiting value of the integral over the arc.

SOLUTION:

a) The magnetic field along the axis is given in Griffiths Example 5.6 (Equation 5.38).

$$B_z(z) = \frac{\mu_0 I}{2} \frac{a^2}{(a^2 + z^2)^{3/2}}$$
(7.1)

$$\int_{-\infty}^{\infty} B_z \, dz = \frac{\mu_0 I}{2} \int_{-\infty}^{\infty} \frac{a^2}{(a^2 + z^2)^{3/2}} \, dz \tag{7.2}$$

Eq. (??) can be evaluated with the change of variables $z = a \tan \theta$, $dz = a \sec^2 \theta d\theta$.

$$\int_{-\infty}^{\infty} B_z \, dz = \frac{\mu_0 I}{2} \int_{-\pi/2}^{\pi/2} \frac{a^3 \sec^2 \theta}{a^3 (1 + \tan^2 \theta)^{3/2}} \, d\theta \tag{7.3}$$

$$=\frac{\mu_0 I}{2} \int_{-\pi/2}^{\pi/2} \cos\theta \,d\theta \tag{7.4}$$

$$\int_{-\infty}^{\infty} B_z \, dz = \mu_0 I \tag{7.5}$$

b) The current enclosed by this path is I. By Ampère's law, this integral should be $\mu_0 I$. We can show this by breaking the integral into two parts; one along the axis and another around the semicircular arc at $r = \infty$.

$$\oint \vec{B} \cdot d\vec{\ell} = \lim_{r \to \infty} \left[\int_{-r}^{r} B_z \, dz + \int_{arc} \vec{B} \cdot d\vec{\ell} \right]$$
(7.6)

$$\int_{-\infty}^{\infty} B_z \, dz + \lim_{r \to \infty} \int_{arc} \vec{B} \cdot d\vec{\ell} \tag{7.7}$$

Now note that \vec{B} goes as $1/r^3$ at $r \gg a$ since the current source has a dipole term. The integral path has length proportional to r. The integral $\int_{\text{arc}} \vec{B} \cdot \vec{\ell}$ has leading term $O(1/r^2)$. In the limit $r \to \infty$, this portion of the integral vanishes.

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I \tag{7.8}$$

8. Current sheet

Griffiths 5.27 (previously 5.26) Find the vector potential above and below the x-y plane with uniform surface current density $\mathbf{K} = k\hat{\mathbf{x}}$.

SOLUTION: Example 5.8 gives the magnetic field for the plane surface current.

$$\vec{B} = \begin{cases} +(\mu_0/2)K\hat{\mathbf{y}} & \text{for } z < 0\\ -(\mu_0/2)K\hat{\mathbf{y}} & \text{for } z > 0 \end{cases}$$
(8.1)

Because the current source is infinite, we can't just integrate it to get the vector potential as would be suggested by Griffiths Equation 5.64. However, we can use the properties of the vector potential to get at it. First, the vector potential generally points in the direction of the current so we would expect $\vec{A} = A_x \hat{\mathbf{x}}$. Also, $\nabla \times \vec{A} = \vec{B}$, so we can use Eq. (??) to derive a differential equation for A_x .

$$\nabla \times \vec{A} = \frac{\partial A_x}{\partial z} \hat{\mathbf{y}} - \frac{\partial \mathbf{A}_x}{\partial \mathbf{y}} \hat{\mathbf{z}} = \tilde{\mathbf{B}}$$
(8.2)

Below the plane we get:

$$\frac{\partial A_x}{\partial z} = \frac{\mu_0 K}{2} \quad \frac{\partial A_x}{\partial y} = 0 \tag{8.3}$$

$$\Rightarrow A_x = \frac{\mu_0 K z}{2} \tag{8.4}$$

Above the plane we have something similar:

$$\frac{\partial A_x}{\partial z} = -\frac{\mu_0 K}{2} \quad \frac{\partial A_x}{\partial y} = 0 \tag{8.5}$$

$$\Rightarrow A_x = -\frac{\mu_0 K z}{2} \tag{8.6}$$

The vector potential is

$$\vec{A} = -\frac{\mu_0 K}{2} |z| \hat{\mathbf{x}} \tag{8.7}$$

We can check this solution by taking its curl and seeing that it gives the expression for the magnetic field. This expression for \vec{A} is not unique, since we can add to it the gradient of any function and its curl will not change.