Physics 3323, Fall 2016 Problem Set 8 *due Oct 21, 2014*

Reading: Griffiths Chapter 5, 6.1-6.2

1. Charged spinning shell

Griffiths 5.37b (previously 5.36) Find the magnetic dipole moment of a spherical shell of radius R spinning with frequency ω, with uniform surface charge density σ. Show that for points $r > R$ the potential is that of a perfect dipole.

SOLUTION:
Suppose that the sphere is spinning about its z-axis. The ring at polar angle θ has charge

$$dq = 2\pi R \sin\theta (R d\theta)\sigma.$$

The current of the ring is

$$dI = dq\frac{\omega}{2\pi}.$$

The area of the ring is

$$A = \pi(R\cos\theta)^2$$

The magnetic moment of the sphere is

$$m = \int (\pi R^2 \sin^2\theta)(2\pi R^2 \sin\theta)\frac{\omega}{2\pi}\sigma d\theta = \frac{4\pi R^4 \omega\sigma}{3}.$$

2. A vector potential

Consider the vector potential $\vec{A}(\vec{r}) = \frac{1}{2}\vec{c} \times \vec{r}$, where $\vec{c}$ is a constant vector.

a) Does this potential satisfy the gauge choice $\vec{\nabla} \cdot \vec{A} = 0$?

b) What is the magnetic field?

SOLUTION:

a) We can use the vector product rules found in the front cover of Griffiths to evaluate $\nabla \cdot \vec{A}$

$$\nabla \cdot \vec{A} = \nabla \cdot \left(\frac{1}{2}\vec{c} \times \vec{r}\right) = \frac{1}{2}\vec{r}(\nabla \times \vec{c}) - \frac{1}{2}\vec{c}(\nabla \times \vec{r}) \tag{2.1}$$

Now $\nabla \times \vec{c} = 0$ since $\vec{c}$ is a constant vector and you can check that $\nabla \times \vec{r} = 0$. Yes, this potential satisfies the gauge choice $\vec{\nabla} \cdot \vec{A} = 0$

b) To get the magnetic field, we just need to take the curl of $\vec{A}$.

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{2}\vec{\nabla} \times (\vec{c} \times \vec{r}) \tag{2.2}$$

$$= \frac{1}{2}\left[(\vec{r} \cdot \vec{\nabla})\vec{c} - (\vec{c} \cdot \vec{\nabla})\vec{r} + \vec{c}(\vec{\nabla} \cdot \vec{r}) - \vec{r}(\vec{\nabla} \cdot \vec{c})\right] \tag{2.3}$$

Now the first and last terms in Eq. (2.3) are both 0 since $\vec{c}$ is a constant vector. The third term is $3\vec{c}$ since $\vec{\nabla} \cdot \vec{r} = 3$. The second term can be calculated as follows:

$$(\vec{c} \cdot \vec{\nabla})\vec{r} = \left(c_x\frac{\partial}{\partial x} + c_y\frac{\partial}{\partial y} + c_z\frac{\partial}{\partial z}\right)(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \tag{2.4}$$

$$= c_x\hat{\mathbf{x}} + c_y\hat{\mathbf{y}} + c_z\hat{\mathbf{z}} = \vec{c} \tag{2.5}$$

Eq. (2.3) is

$$\boxed{\vec{B} = \frac{1}{2}[-\vec{c} + 3\vec{c}] = \vec{c}} \tag{2.6}$$

3. Vector potential of infinite solenoid

You have an infinitely long solenoid with radius R and N turns of wire per unit length. The wire carries a current I.

a) Find the vector potential of this solenoid. Avoid looking up the answer. It is useful to begin by showing that $\oint \vec{A}\cdot d\vec{\ell} = \int \vec{B}\cdot \hat{n} dA$, where the LHS integral is around the edge of an open surface with area A.

b) Compute $\vec{\nabla}\times\vec{A}$ in cylindrical coordinates to show that you get the correct magnetic field.

SOLUTION:

a) We know the magnetic field in the solenoid (Griffiths Example 5.9).

$$\vec{B} = \begin{cases} \mu_0 nI\,\hat{\mathbf{z}} & \text{inside} \\ 0 & \text{outside} \end{cases} \tag{3.1}$$

Let's start by showing $\oint \vec{A}\cdot d\vec{\ell} = \int \vec{B}\cdot\hat{n}dA$ as per the hint. This is an application of Stokes theorem to the definition of the vector potential

$$\int \vec{B}\cdot\hat{n}\,dA = \int \left(\vec{\nabla}\times\vec{A}\right)\cdot\hat{n}\,dA = \oint \vec{A}\cdot d\vec{\ell} \tag{3.2}$$

Now the vector potential generally points in the direction of the current, so let's suppose that $\vec{A} = A_\phi\hat{\phi}$, that is the vector potential points circumferentially. We create circumferential Ampèrian loops. Inside the solenoid Eq. (3.2) becomes the following:

$$\mu_0 nI\,\pi s^2 = 2\pi s A_\phi \;\;\Rightarrow\;\; A_phi = \frac{\mu_0 nIs}{2} \tag{3.3}$$

Outside the solenoid we get:

$$\mu_0 nI\,\pi R^2 = 2\pi s A_\phi \;\;\Rightarrow\;\; A_phi = \frac{\mu_0 nIR^2}{2s} \tag{3.4}$$

The solution is:

$$\boxed{\vec{A} = \begin{cases} \frac{\mu_0 nIs}{2}\,\hat{\phi} & \text{inside} \\ \frac{\mu_0 nIR^2}{2s}\,\hat{\phi} & \text{outside} \end{cases}} \tag{3.5}$$

b) We didn't justify why the vector potential should point circumferentially in the last part. The only justification we need is to see if taking the curl of the vector potential gives the correct magnetic field.

$$\vec{B} = \vec{\nabla}\times\vec{A} = -\frac{\partial A_\phi}{\partial z}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial}{\partial s}(sA_\phi)\,\hat{\mathbf{z}} \tag{3.6}$$

Inside:

$$\boxed{\vec{B} = \frac{1}{s}\frac{\partial}{\partial s}\left(\frac{\mu_0 nIs^2}{2}\right)\hat{\mathbf{z}} = \mu_0 nI\,\hat{\mathbf{z}}} \tag{3.7}$$

Outside:

$$\boxed{\vec{B} = \frac{1}{s}\frac{\partial}{\partial s}\left(\frac{\mu_0 nIR^2}{2}\right)\hat{\mathbf{z}} = 0} \tag{3.8}$$

Comparison with Eq. (3.1) shows that this is indeed a correct vector potential.

4. Square current loop

Griffiths 5.36. Find the exact magnetic field a distance d above the center of a square loop of side w, carrying a current I. Verify that it reduces to the field of a dipole with the appropriate dipole moment, when $z \gg w$.

SOLUTION:

Place the loop in the x-y plane with its center at the origin.

We want the field at $z = d$. The magnetic field from a current distribution is

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^{3/2}}$$

Consider the contribution from the sides that extends from $x = -w/2$ to $x = w/2$ at $y = w/2$.

$$d\mathbf{l}' = \hat{\mathbf{x}} dx$$

$$\mathbf{r} = d\hat{\mathbf{z}}$$

$$\mathbf{r}' = x\hat{\mathbf{x}} + \frac{w}{2}\hat{\mathbf{y}}$$

Then

$$d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}') = (-\frac{w}{2}\hat{\mathbf{z}} + d\hat{\mathbf{y}})$$

Next

$$|\mathbf{r} - \mathbf{r}'|^3 = (x^2 + (w/2)^2 + d^2)^{3/2}$$

Putting the pieces together we have the contribution to the z-component of the field from one size is

$$B_z(z = d) = \frac{\mu_0}{4\pi} I \int_{w/2}^{-w/2} \frac{\frac{-w}{2} dx}{(x^2 + (w/2)^2 + d^2)^{3/2}}$$

$$= \frac{\mu_0 I}{8\pi} \frac{w^2}{((w/2)^2 + d^2)(w^2/2 + d^2)^{1/2}}$$

Each side will contribute the same. The total field in the z-direction will be

$$B_z = \frac{\mu_0 I}{2\pi} \frac{w^2}{((w/2)^2 + d^2)(w^2/2 + d^2)^{1/2}}$$

The net y-component of the field will be zero by symmetry. In the limit $z \gg w$

$$B_z = \frac{\mu_0 I}{2\pi} \frac{w^2}{d^3}$$

The field of a dipole is

$$\mathbf{B}_{dip} = \frac{\mu_0}{4\pi} \frac{m}{r^3} (2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta})$$

At $r = d, \theta = 0$, and $m = w^2 I$,

$$\mathbf{B}_{dip} = \frac{\mu_0}{2\pi} \frac{w^2 I}{d^3} \hat{\mathbf{r}}$$

5. Magnetized cylinder

Griffiths 6.12. An infinitely long cylinder, of radius R, carries a "frozen-in" magnetization, parallel to the axis,

$$\mathbf{M} = ks\hat{\mathbf{z}},$$

where k is a constant and s is the distance from the axis; there is no free current anywhere. Find the magnetic field inside and outside the cylinder by two different methods:

1. As in Section 6.2, locate all of the bound currents, and calculate the field they produce.
2. Use Ampere's law (in the form of Eq. 6.20) to find **H**, and then get **B** from Eq. 6.18. (Notice that the second method is much faster, and avoids any explicit reference to the bound currents.)

SOLUTION:

a) There can be bound current density in the cylinder and a bound surface current at the boundary of the cylinder $s = R$.

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = \frac{1}{s}\frac{\partial M_z}{\partial \phi}\,\hat{\mathbf{s}} - \frac{\partial M_z}{\partial s}\,\hat{\phi} = -k\,\hat{\phi} \qquad (5.1)$$

$$\vec{K}_b = \vec{M} \times \hat{n} = kR\,\hat{\mathbf{z}} \times \hat{\mathbf{s}} = kR\,\hat{\phi} \tag{5.2}$$

We can think of cylindrical shells of thickness dz inside the cylinder. If we divide up the cylinder like this, then we have a set of concentric solenoids with surface current $\vec{J}_b\,ds$ plus a solenoid with surface current $\vec{K}_b$ at the outside. The magnetic field at any point will have contributions from each of the solenoids that surrounds it. Just as a reminder, a solenoid with surface current $K\,\hat{\phi}$ has magnetic field $B = \mu_0 K\,\hat{\mathbf{z}}$ inside. The magnetic field outside the cylinder is 0. Inside the cylinder we have:

$$\vec{B} = \mu_0 K_b\,\hat{\mathbf{z}} + \mu_0\,\hat{\mathbf{z}} \int_s^R J_b\,ds = \mu_0 kR\,\hat{\mathbf{z}} - \mu_0\,\hat{\mathbf{z}} \int_s^R k\,ds \tag{5.3}$$

$$\boxed{\vec{B} = (\mu_0 kR - \mu_0 kR + \mu_0 ks)\,\hat{\mathbf{z}} = \mu_0 ks\,\hat{\mathbf{z}}} \tag{5.4}$$

b) Here we just use Ampère's law. $\oint \vec{H} \cdot d\vec{\ell} = I_{fenc}$. Since there is no free current anywhere, $\vec{H} = 0$ everywhere. Now we use the relation:

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} \tag{5.5}$$

$$\boxed{\vec{B} = \mu_0 \vec{M} = \mu_0 ks\,\hat{\mathbf{z}}} \tag{5.6}$$