Physics 3323, Fall 2016 Problem Set 9 *due October 28, 2016*

Reading: Griffiths 7.1–7.2.2

1. Paramagnetic sample

A cylindrical solenoid has a single layer winding of radius r_0. It is so long that near one end, the field may be taken to be that of a semi-inifinite solenoid. Show that the point on the axis of the solenoid where a small paramagnetic sample will experience the greatest force is located at a distance $r_0/\sqrt{15}$ in from the end.

SOLUTION: The first thing we need to do is find out what the magnetic field from a semi-infinite solenoid is along the axis. To do this, we set up the semi-infinite solenoid so that it starts at $z = 0$ and extends to $z = \infty$. We don't really know about a semi-infinite solenoid, but we do know the field along the axis due to a ring of current centered on the axis. Griffiths Example 5.6.

$$B_z = \frac{\mu_0 I}{2} \frac{r_0^2}{(r_0^2 + z^2)^{3/2}} \tag{1.1}$$

Now since we know the field along the axis due to a ring of current centered on the axis, we can slice the solenoid into a bunch of rings centered on the z-axis at different values of z' and add the contributions from each. The current in each ring will be the current in the solenoid multiplied by the number of turns in the ring. $I_{\text{ring}} = Indz'$, where n is the turn density of the solenoid. Since this will be the contribution from the free current only (not the induced dipole moment in the paramagnetic material), it will be the $\vec{H} = \vec{B}/\mu_0$ field, not the total magnetic field $\vec{B}$. The $\vec{H}$ field at z due to a ring at z' will be given as:

$$dH_z = \frac{In\,dz'}{2} \frac{r_0^2}{\left(r_0^2 + (z'-z)^2\right)^{3/2}} \tag{1.2}$$

Now we can add up the contribution from all of the rings.

$$H_z = \frac{In}{2} \int_0^\infty \frac{r_0^2}{\left(r_0^2 + (z'-z)^2\right)^{3/2}}\, dz' \tag{1.3}$$

The way to do this integral is by a change of variables $z' - z = r_0 \tan\theta$, $dz' = r_0 \sec^2\theta\, d\theta$.

$$H_z = \frac{In}{2} \int_{\theta_0}^{\theta_\infty} \frac{r_0^3 \sec^2\theta}{r_0^3 \sec^3\theta}\, d\theta = \frac{In}{2} \int_{\theta_0}^{\theta_\infty} \cos\theta\, d\theta \tag{1.4}$$

$$H_z = \frac{In}{2} \sin\theta \bigg|_{\theta_0}^{\theta_\infty} = \frac{In}{2} \frac{z'-z}{\sqrt{r_0^2 + (z'-z)^2}} \bigg|_0^\infty \tag{1.5}$$

$$\boxed{H_z = \frac{In}{2} \left[1 + \frac{z}{\sqrt{r_0^2 + z^2}}\right]} \tag{1.6}$$

This solution makes sense in the limits $z \to \pm\infty$ since it goes to the field due to an infinite solenoid when $z \to +\infty$ while it goes to zero as $z \to -\infty$.

Now that we know the $\vec{H}$ field due to the semi-infinite solenoid we can calculate the induced dipole moment in the small piece of paramagnetic material. The material will have some (positive) magnetic susceptibility χ_m. The induced magnetization will be $\vec{M} = \chi_m \vec{H}$. The magnetization is the dipole moment per unit volume, so if the volume of the piece of material is V, the total induced dipole moment is $\vec{m} = V\vec{M} = V\chi_m\vec{H}$.

The force on a dipole in an external magnetic field is $\vec{\nabla}(\vec{m}\cdot\vec{B}_{\text{ext}})$. Here $\vec{B}_{\text{ext}}$ is just the external field (not including the field due to the induced dipole). In our case it is the magnetic field from the semi-infinite solenoid $\vec{B}_{\text{ext}} = \mu_0\vec{H}$

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B}_{\text{ext}}) = \vec{\nabla}\left(V\chi_m\vec{H}\cdot\mu_0\vec{H}\right) \tag{1.7}$$

In our case (along the z-axis) the fields are only in the z direction and only depend on z, so this reduces to a one-dimensional problem.

$$F = V\chi_m\mu_0\frac{d}{dz}\left(H_z^2\right) \tag{1.8}$$

$$= V\chi_m\mu_0\frac{I^2n^2}{4}\frac{d}{dz}\left(1+\frac{z}{\sqrt{r_0^2+z^2}}\right)^2 \tag{1.9}$$

Now before we evaluate the derivative, note that we want the maximum in the force so we will need to take a second derivative, set it equal to zero, and solve for z. The constants will pass through this derivative opperation and can be divided out. I want to solve the following equation for z.

$$\frac{d^2}{dz^2}\left(1+\frac{z}{\sqrt{r_0^2+z^2}}\right)^2 = 0 \tag{1.10}$$

To simplify this a little, let $T = T(z) = \frac{z}{\sqrt{r_0^2+z^2}}$.

$$\frac{d^2}{dz^2}(1+T)^2 = \frac{d}{dz}\left[2(1+T)\frac{dT}{dz}\right] = 0 \tag{1.11}$$

$$\frac{d}{dz}\left[(1+T)\frac{dT}{dz}\right] = \left(\frac{dT}{dz}\right)^2 + (1+T)\frac{d^2T}{dz^2} = 0 \tag{1.12}$$

Now I can calculate the derivatives of T one at a time.

$$\frac{dT}{dz} = \frac{d}{dz}\left(\frac{z}{\sqrt{r_0^2+z^2}}\right) = \frac{\sqrt{r_0^2+z^2} - z\frac{1}{2\sqrt{r_0^2+z^2}}2z}{r_0^2+z^2} \tag{1.13}$$

$$\frac{dT}{dz} = \frac{1}{\sqrt{r_0^2+z^2}} - \frac{z^2}{\left(r_0^2+z^2\right)^{3/2}} \tag{1.14}$$

$$\boxed{\frac{dT}{dz} = \frac{1}{z}\left(T - T^3\right)} \tag{1.15}$$

Now for the second derivative:

$$\frac{d^2T}{dz^2} = \frac{d}{dz}\frac{dT}{dz} = \frac{d}{dz}\left(\frac{1}{z}\left(T-T^3\right)\right) \tag{1.16}$$

$$\frac{d^2T}{dz^2} = \frac{z\left(1-3T^2\right)\frac{dT}{dz} - \left(T-T^3\right)}{z^2} \tag{1.17}$$

$$\boxed{\frac{d^2T}{dz^2} = -\frac{3T^2}{z^2}\left(T-T^3\right)} \tag{1.18}$$

Now we have the derivatives and we can insert them into Eq. (1.12).

$$\left(\frac{dT}{dz}\right)^2 + (1+T)\frac{d^2T}{dz^2} = \frac{T^2}{z^2}\left(1-T^2\right)^2 - (1+T)\frac{T^3}{z^2}\left(1-T^2\right) \tag{1.19}$$

This factors nicely.

$$= \frac{T^2}{z^2}(1+T)^2(1-T)(1-4T) = 0 \tag{1.20}$$

You can check that setting $T = 0$ gives an inflection point at $z = 0$ while $1 \pm T = 0$ gives $z = \pm\infty$ respectively. Setting $1 - 4T = 0$ gives:

$$1 - \frac{4z}{\sqrt{r_0^2+z^2}} = 0 \quad\Rightarrow\quad 4z = \sqrt{r_0^2+z^2} \tag{1.21}$$

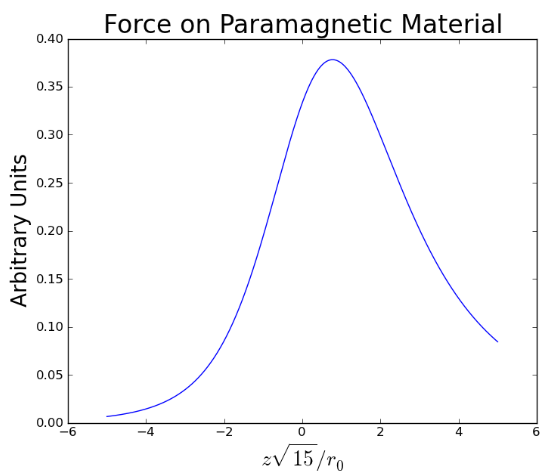


Figure 1: Note that z is measured in units of $r_0/\sqrt{15}$

$$16z^2 = r_0^2 + z^2 \quad \Rightarrow \quad z^2 = \frac{r_0^2}{15} \quad \Rightarrow \quad z = \pm\frac{r_0}{\sqrt{15}} \tag{1.22}$$

A quick plot of a quantity proportional to F, $((1+T)\frac{dT}{dz})$, shows that the negative solution is also a saddle point and that the positive solution is a maximum.

$$\boxed{z = \frac{r_0}{\sqrt{15}}} \tag{1.23}$$

2. Sphere of magnetic material in uniform B field

a) Griffiths 6.18. A sphere of linear magnetic material is placed in an otherwise uniform magnetic field $\mathbf{B_0}$. Find the new field inside the spere. [$Hint$: See Prob. 6.15 or Prob. 4.23.]

b) Griffiths 6.25 (was 6.23), part b. Notice the following parallel:

$$\begin{cases} \nabla\cdot\mathbf{D}=0, & \nabla\times\mathbf{E}=0, \quad \epsilon_0\mathbf{E}=\mathbf{D}-\mathbf{P}, \quad \text{(no free charge.)} \\ \nabla\cdot\mathbf{B}=0, & \nabla\times\mathbf{H}=0, \quad \mu_0\mathbf{H}=\mathbf{B}-\mu_0\mathbf{M}, \quad \text{(no free current.)} \end{cases}$$

Thus the transcription $\mathbf{D}\to\mathbf{B}, \mathbf{E}\to\mathbf{H}, \mathbf{P}\to\mu_\mathbf{0}\mathbf{M}, \epsilon_0 \to \mu_0$ turns an electrostatic problem into an analogous magnetostatic one. Use this, together with your knowledge of the electrostatic results, to rederive the magnetic field inside a sphere of linear magnetic material in an otherwise uniform magnetic field (Prob. 6.18).

SOLUTION:

a) We'll let the otherwise uniform magnetic field point in the z direction. $\vec{B}_0 = B_0\,\hat{\mathbf{z}}$. Since $\vec{J}_f = 0$ everywhere, we can solve this in the form of Griffiths problem 6.15 by using the magnetic scalar potential where $\vec{H} = -\vec{\nabla}W$. We can expand the magnetic scalar potential in terms of Legendre polynomials.

$$W = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta) \tag{2.1}$$

Since the inside of the sphere is different from the outside, we'll have a solution for outside the sphere W_{out} and a solution for inside the sphere W_{in}. These will fit the following boundary conditions.

1. $$W_{\text{out}} \to -\frac{B_0}{\mu_0}z = -\frac{B_0}{\mu_0}r\cos\theta \text{ as } r\to\infty$$

2. $$W_{\text{out}}(R) = W_{\text{in}}(R)$$

3.

$$\mu_0 \left.\frac{\partial W_{\text{out}}}{\partial r}\right|_R = \mu \left.\frac{\partial W_{\text{in}}}{\partial r}\right|_R$$

4.

$$W_{\text{in}}(0) < \infty$$

These boundary conditions should be sufficient to determine W. The first comes from the fact that at large r, the field should go to the uniform field $B_0\,\hat{\mathbf{z}}$. This means that $\vec{H} = \frac{1}{\mu_0}\vec{B} \to \frac{B_0}{\mu_0}\,\hat{\mathbf{z}}$, which leads to boundary condition 1. Boundary conditions 2 and 3 are the regular magnetostatic boundary conditions.

Boundary conditions 1 and 4 yield:

$$W_{\text{out}} = -\frac{B_0}{\mu_0} r\cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \tag{2.2}$$

$$W_{\text{in}} = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \tag{2.3}$$

Boundary condition 2 gives:

$$A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}} \quad \Rightarrow \quad B_\ell = A_\ell R^{2\ell+1} \quad \text{for} \quad \ell \neq 1 \tag{2.4}$$

$$A_1 R = -\frac{B_0}{\mu_0} R + \frac{B_1}{R^2} \quad \Rightarrow \quad B_1 = \left(A_1 + \frac{B_0}{\mu_0}\right) R^3 \tag{2.5}$$

Boundary condition 3 gives:

$$\mu\ell A_\ell R^{\ell-1} = -\mu_0(\ell+1)\frac{B_\ell}{R^{\ell+2}}$$

$$\Rightarrow B_\ell = \frac{\mu\ell}{\mu_0(l+1)} A_\ell R^{2\ell+1} \quad \text{for} \quad \ell \neq 1 \tag{2.6}$$

$$\mu A_1 = -\left(B_0 + \frac{2\mu_0 B_1}{R^3}\right) \quad \Rightarrow \quad B_1 = -\left(\frac{\mu A_1 + B_0}{2\mu_0}\right) R^3 \tag{2.7}$$

Equating Eq. (2.4) and Eq. (2.6) gives:

$$A_\ell R^{2\ell+1} = A_\ell R^{2\ell+1} \frac{\mu\ell}{\mu_0(\ell+1)} \quad \Rightarrow \quad A_\ell = 0 \quad \text{for} \quad \ell \neq 1 \tag{2.8}$$

EquatingEq. (2.5) and Eq. (2.7) gives:

$$\left(A_1 + \frac{B_0}{\mu_0}\right) R^3 = -\left(\frac{\mu A_1 + B_0}{2\mu_0}\right) R^3$$

$$\Rightarrow \quad A_1 = -\frac{3B_0}{2\mu_0 + \mu} \tag{2.9}$$

The magnetic scalar potential inside the sphere is

$$W_{\text{in}} = -\frac{3B_0}{2\mu_0 + \mu} r\cos\theta = -\frac{3B_0}{2\mu_0 + \mu} z \tag{2.10}$$

This gives the magnetic field in the sphere.

$$\boxed{\vec{B} = \mu\vec{H} = -\mu\vec{\nabla}W_{\text{in}} = \frac{3\mu B_0}{2\mu_0 + \mu}\,\hat{\mathbf{z}} = \frac{3\mu}{2\mu_0 + \mu}\vec{B}_0} \tag{2.11}$$

In terms of the relative permeability we have:

$$\boxed{\vec{B} = \frac{3\mu_r}{2 + \mu_r}\vec{B}_0} \tag{2.12}$$

b) Griffiths Example 4.7 gives the formula for a uniform dielectric sphere in an otherwise uniform electric field.

$$\vec{E} = \frac{3}{\varepsilon_r + 2}\vec{E}_0 \tag{2.13}$$

Using the prescribed transformation, we get:

$$\vec{H} = \frac{3}{\mu_r + 2}\vec{H}_0 \tag{2.14}$$

Now $\vec{H}_0$ is defined for outside the sphere so $\vec{H}_0 = \vec{B}_0/\mu_0$ and $\vec{H}$ is describing the inside of the sphere, so $\vec{H} = \vec{B}/\mu$. The field in the sphere is:

$$\boxed{\vec{B} = \frac{3\mu/\mu_0}{\mu_r + 2}\vec{B}_0 = \frac{3\mu_r}{\mu_r + 2}\vec{B}_0} \tag{2.15}$$

We see that this is the same as Eq. (2.12).

3. Hemispherical capacitor

The hemispherical conducting shell (radius b) of Problem Set 6 is filled with a soft conducting plastic, characterized by a very large relative dielectric constant ε_r and a very small electrical conductivity σ. A needle-shaped conductor with a hemispherical tip (radius a) is pressed into the plastic, as shown in the figure, so that it is concentric with the shell.

a) If the conductors are maintained at a potential difference V_0, what current I flows from one to the other?

b) What is the resistance of this arrangement?

c) You now disconnect the battery so the charge will gradually leak off. Show that $V(t) = V_0 e^{-t/\tau}$, and find the time constant τ in terms of the permittivity and conductivity.

SOLUTION:

a) From Problem Set 6 we have the electric field in the capacitor.

$$\vec{E} = \frac{Q}{4\pi\varepsilon r^2}\hat{\mathbf{r}} \tag{3.1}$$

From this the current density is:

$$\vec{J} = \sigma\vec{E} = \frac{Q\sigma}{4\pi\varepsilon r^2}\hat{\mathbf{r}} \tag{3.2}$$

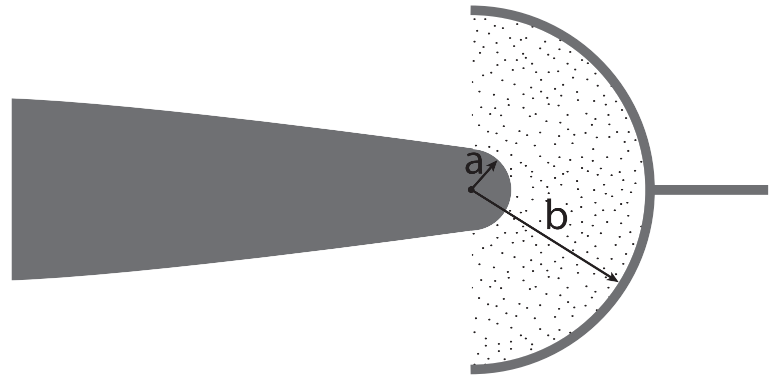


Figure 2: Capacitor for problem 3.

Now we would like to know Q in terms of V_0. This is just $Q = CV_0$ and we know C from problem set 6.

$$C = 2\pi\varepsilon\left(\frac{ab}{b-a}\right) \tag{3.3}$$

$$\vec{J} = \frac{V_0\sigma 2\pi\varepsilon}{4\pi\varepsilon r^2}\left(\frac{ab}{b-a}\right)\hat{\mathbf{r}} = \frac{V_0\sigma}{2r^2}\left(\frac{ab}{b-a}\right)\hat{\mathbf{r}} \tag{3.4}$$

Now we can set up a hemispherical surface at some radius R and integrate the current density over it to get the current

$$I = \int \vec{J}\cdot d\vec{A} = \frac{V_0\sigma}{2R^2}\left(\frac{ab}{b-a}\right)\int dA = \frac{V_0\sigma}{2R^2}\left(\frac{ab}{b-a}\right)2\pi R^2 \tag{3.5}$$

$$\boxed{I = V_0\sigma\pi\left(\frac{ab}{b-a}\right)} \tag{3.6}$$

This is independent of the particular radius R we chose to integrate over, as it should be.

b)

$$R = \frac{V_0}{I} \tag{3.7}$$

$$\boxed{R = \frac{1}{\pi\sigma}\left(\frac{1}{a} - \frac{1}{b}\right)} \tag{3.8}$$

c) The current is the rate at which the charge on the capacitor is leaving, so $I = -\frac{dQ}{dt}$.

$$Q = CV = CRI = -RC\frac{dQ}{dt} \tag{3.9}$$

The solution to this differential equation is:

$$Q = Q_0 e^{-t/RC} \tag{3.10}$$

In terms of $V = Q/C$ we have

$$\boxed{V = \frac{Q}{C} = \frac{Q_0}{C}e^{-t/\tau} = V_0 e^{-t/\tau}} \tag{3.11}$$

$$\boxed{\tau = RC = \frac{1}{\pi\sigma}\left(\frac{1}{a} - \frac{1}{b}\right)2\pi\varepsilon\left(\frac{1}{a} - \frac{1}{b}\right)^{-1} = \frac{2\varepsilon}{\sigma}} \tag{3.12}$$

4. Susceptible wire

Griffiths 6.17. A current I flows down a long straight wire of radius a. If the wire is made of linear material (copper, say or aluminum) with susceptibility χ_m, and the current is distributed unifromly, what is the magnetic field a distance s from the axis? Find all the bound currents. What is the net bound current flowing down the wire?

SOLUTION: **H** is given in terms of the free current

$$\int \mathbf{H}(s) \cdot d\mathbf{l} = \frac{s^2}{a^2}I \to \mathbf{H}(s) = \frac{s}{2\pi a^2}I\hat{\phi}$$

$$\mathbf{B} = \mu_\mathbf{0}(\mathbf{1} + \chi_\mathbf{m})\mathbf{H}$$

Then

$$\mathbf{M}(s) = -\mathbf{H} + \frac{1}{\mu_0}\mathbf{B} = \chi_\mathbf{m}\mathbf{H}$$

The bound current

$$\mathbf{J}_b = \nabla \times \mathbf{M} = \frac{1}{s}\frac{\partial}{\partial s}(sM_\phi)\hat{\mathbf{z}} = \frac{\chi_m}{\pi a^2}I\hat{\mathbf{z}}$$

The surface bound current

$$\mathbf{K}_s = \mathbf{M} \times \hat{\mathbf{n}} = -\frac{\chi_m}{2\pi a}I\hat{\mathbf{z}}$$

5. Space charge effect

We'll look at conducting plates of area A separated by a distance s, with a high enough electron density to affect the field between the plates. In this limit we are interested in, the "space charge" from the electron density near the cathode large enough to cancel the field from the plates in that region. Let's summarize the facts for this limit:

- $\vec{J} \cdot \vec{A} = I$ (see Fig. 1),
- $V(x) = 0$ and $dV/dx = 0$ at $x = 0$,
- $V(x) = V_0$ at $x = s$,
- Poisson: $d^2V/dx^2 = -\rho/\varepsilon_0$,
- $\rho v(x) = J_x = -I/A$,
- conservation of energy: $mv^2/2 = eV(x)$,

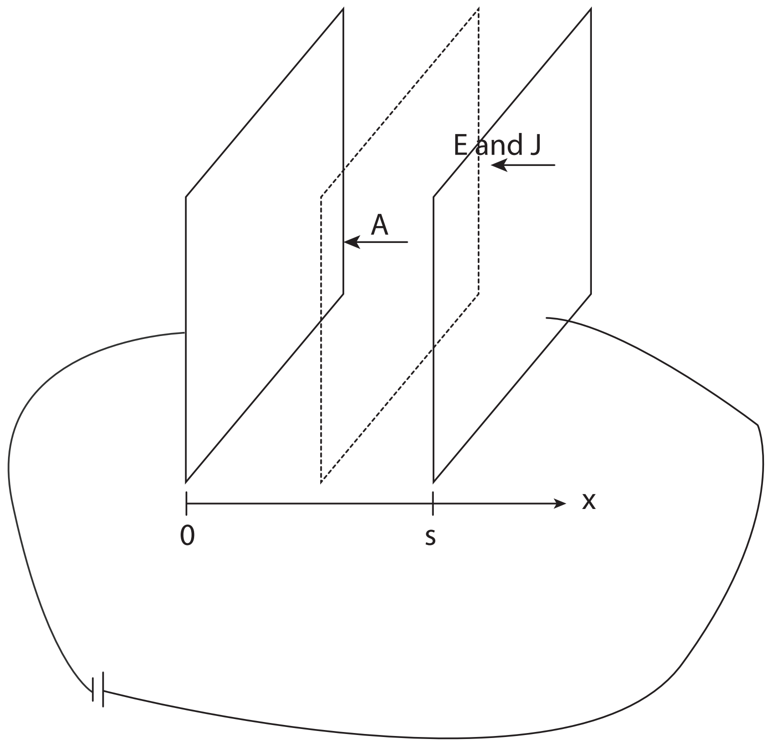


Figure 3: Geometry for problem 5.

which will provide all the ingredients to solve for the potential as a function of position (and hence for the electric field, charge density, etc.).

a) Show that the potential satisfies

$$\frac{d^2V}{dx^2} = KV^{-1/2}$$

and find the constant K.

b) You can solve this differential equation by multiplying both sides by the function $2\,dV/dx$, integrating, and performing a (different) change of integration variable on each side. By doing so, find an equation for $(dV/dx)^2$, using the boundary conditions to determine any constants of integration.

c) Taking the square root gives you a first order differential equation for V. By rearranging the terms to isolate V and integrating, show that $V \propto x^{4/3}$. Sketch V and the magnitude of the electric field as a function of distance into the gap.

d) Show that the relationship between the applied voltage difference and the resulting current across the gap is given by $V_0^{3/2} \propto I$, and hence that this configuration does not satisfy Ohm's law.

SOLUTION:

a) From Poisson, definition of current and conservation of energy

$$\frac{dV^2}{dx^2} = -\rho/\epsilon_0, \quad mv^2/2 = eV(x), \quad \text{and } \rho(x) = -I/A$$

we get

$$\frac{dV^2}{dx^2} = -\frac{-I}{vA} = \frac{I}{A}\sqrt{\frac{m}{2eV}} = KV^{-1/2}$$

b) Multiply both sides by $2dV/dx$

$$2\frac{d^2V}{dx^2}\frac{dV}{dx} = 2KV^{-1/2}\frac{dV}{dx}$$

$$2\frac{dW}{dx}W = 2KV^{-1/2}\frac{dV}{dx}$$

$$2\int W dW = 2\int KV^{-1/2}dV$$

$$W^2 \mid_0^{W_0} = 4KV^{1/2} \mid_0^{V_0}$$

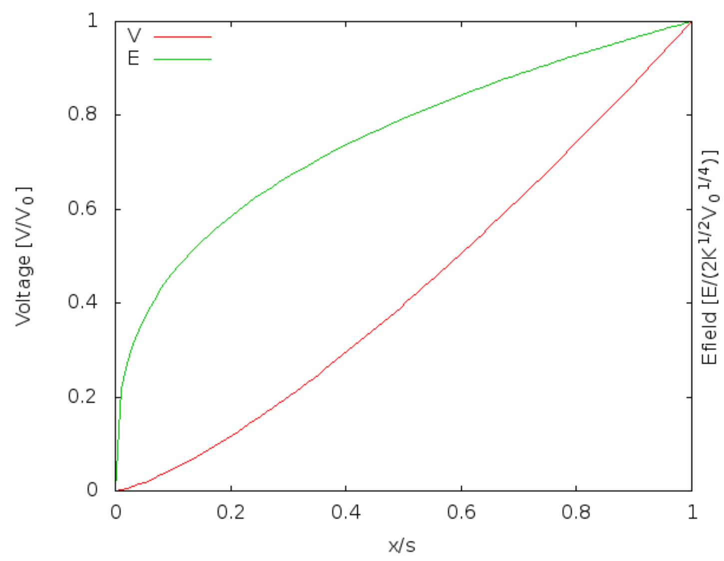
$$W^2(x) = 4KV^{1/2} \rightarrow \frac{dV}{dx} = 2K^{1/2}V^{1/4}$$

c)

$$\frac{dV}{dx} = 2K^{1/2}V^{1/4}$$

$$\int_0^{V_0} V^{-1/4}dV = 2K^{1/2}\int_0^s dx$$

$$\frac{4}{3}V^{3/4} = 2K^{1/2}x \rightarrow V(x) = \left(\frac{3}{2}\right)^{4/3} K^{2/3}x^{4/3}$$



d) At $x = s$ we have that

$$V_0 = \left(\frac{3}{2}\right)^{4/3} K^{2/3}s^{4/3}$$

Now to solve for I first write

$$K = V_0^{3/2}\left(\frac{2}{3}\right)^{4/3} \frac{1}{s^{4/3}}$$

But $K = I\sqrt{\frac{m}{2eV}}\frac{1}{A}$ so

$$V_0^{3/2} \propto I$$

6. Superconducting electrodes

Two plane parallel superconducting electrodes are separated by an Ohm's law medium of thickness s and cross sectional area A. Its resistivity $\rho(z)$ varies linearly from ρ_0, at the positive electrode, to $\rho_0 + a$ at the negative electrode. The permittivity ε is constant. Neglect edge effects and assume a steady current I.

a) **F**ind the total resistance of this Ohm's law medium

b) **F**ind the free charge density in the volume.

c) **F**ind the free surface charge densities on the boundaries.

d) **W**hat is the total charge on the resistor?

SOLUTION:

a) $\mathbf{J} = \mathbf{E}/\rho(\mathbf{z})$.

$$\mathbf{J} \cdot \mathbf{A} = I = \mathbf{E} \cdot \mathbf{A}/(\rho_\mathbf{0} + \mathbf{a}\frac{\mathbf{z}}{\mathbf{s}})\hat{\mathbf{z}} \rightarrow \mathbf{E} = \frac{\mathbf{I}}{\mathbf{A}}(\rho_\mathbf{0} + \mathbf{a}\frac{\mathbf{z}}{\mathbf{s}})\hat{\mathbf{z}}$$

$$\int_0^s \mathbf{E} \cdot \mathbf{dl} = \mathbf{V} = \frac{\mathbf{I}}{\mathbf{A}}(\rho_\mathbf{0}\mathbf{s} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{as})$$

$$\rightarrow R = \frac{V}{I} = \frac{s}{A}(\rho_0 + \frac{1}{2}a).$$

b)

$$\mathbf{D} = \epsilon\mathbf{E}$$

$$\rho_f = \nabla \cdot \mathbf{D} = \epsilon\frac{I}{A}\frac{a}{s}$$

c) $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

$$\sigma_b(z=0) = -\epsilon_0\chi_e\frac{I}{A}\rho_0$$

$$\sigma_b(z=s) = \epsilon_0\chi_e\frac{I}{A}(\rho_0 + a)$$

where $\chi_e = \epsilon/\epsilon_0 - 1$. Also

$$\sigma_f(0) = D_\perp^{above} - D_\perp^{below} = \epsilon\frac{I}{A}\rho_0$$

$$\sigma_f(s) = D_\perp^{above} - D_\perp^{below} = -\epsilon\frac{I}{A}(\rho_0 + a)$$

and

$$\rho_b(z) = -\nabla \cdot \mathbf{P} = -\epsilon_0\chi_e\frac{I}{A}\frac{a}{s}$$

d) The total charge

$$\begin{aligned} Q &= A\int_0^s (\rho_f + \rho_b)dz \\ &= A\int_0^s \left(\epsilon\frac{I}{A}\frac{a}{s} - \epsilon_0\chi_e\frac{I}{A}\frac{a}{s}\right)dz \\ Q &= Ia(\epsilon - \epsilon_0\chi_e) \\ &= Ia\epsilon_0 \end{aligned}$$

7. Earth's field

A method for measuring the earth's magnetic field is to use a flip coil and a ballistic galvanometer. The coil, of radius a, turns quickly through 180 degrees. The galvanometer measures the total charge that flows through the coil as it flips.

a) **E**xplain how the vector $\vec{B}$ can be measured.

b) **S**uppose the coil axis is initially parallel to the magnetic field, which has a strength of 0.5×10^{-4} T,İf the radius $a = 5$ cm, and the resistance of the coil is $R = 0.1\Omega$, what total charge flows through the coil as it flips over?

SOLUTION:

a)Suppose there is some flux $\Phi = BA$. As the coil flips through 180 degrees the change in the flux through the coil $\Delta\Phi = 2\Phi$. If the coil is in series with a resistor and a capacitor, then

$$\mathcal{E}\Delta t = -\frac{d\Phi}{dt}\Delta t = (IR + \frac{Q}{C})\Delta t$$

Measure the charge Q that flowed onto the capcitor to get the flux. Divide by the area of the loop to get the field.

b) Since $I = \mathcal{E}/R$ and

$$Q = \Delta t I = \frac{1}{R}\Delta\Phi = 2\frac{1}{0.1}(0.5\times10^{-4})(\pi(0.05)^2) = 7.8\times10^{-6}\text{Coulombs}$$

8. Sliding bar

Griffiths 7.7. A metal bar of mass m slides frictionlessly on two parallel conducting rails a distance l apart. A resistor R is connected across the rails, and a uniform magnetic field $\mathbf{B}$, pointing into the page, fills the entire region.

a) If the bar moves to the right at speed v, what is the current in the resistor? In what direction does it flow?

b) What is the magnetic force on the bar? In what direction?

c) If the bar starts out with speed v_0 at time $t = 0$, and is left to slide, what is the speed at a later time t?

d) The initial kinetic energy of the bar was, of course, $\frac{1}{2}mv_0^2$. Check that the energy delivered to the resistor is exactly $\frac{1}{2}mv_0^2$.

SOLUTION:

a)The change in flux $\frac{d\Phi}{dt} = lvB = -\mathcal{E} = -RI$

$$\rightarrow I = \frac{lvB}{R}$$

in the clockwise direction.

b)The magnetic force on the bar is due to the current flowing in the circuit.

$$\mathbf{F} = I\int d\mathbf{l}\times\mathbf{B} = \frac{vl^2B^2}{R}$$

to the left.

c)Use $\mathbf{F} = m\mathbf{a}$.

$$\frac{vl^2B^2}{R} = m\frac{dv}{dt}$$

Integration gives us

$$v = v_0 e^{-t/\tau}$$

where $\tau = \frac{Rm}{l^2B^2}$.

d)The total energy dissipated in the resistor is

$$W = \int_0^\infty I^2 R dt$$

$$I(t) = \frac{lv(t)B}{R} = v_0 e^{-t/\tau}\frac{lB}{R}$$

$$W = \left(\frac{lB}{R}\right)^2 v_0^2 R\int_0^\infty e^{-2t/\tau}dt = \left(\frac{lB}{R}\right)^2 v_0^2 R\frac{\tau}{2}$$

Substitution for $\tau = \frac{Rm}{l^2B^2}$ gives

$$W = \frac{1}{2}\left(\frac{lB}{R}\right)^2 v_0^2\frac{Rm}{l^2B^2}R = \frac{1}{2}mv_0^2$$