Intermediate Electricity and Magnetism

August 29-September 2, 2016

Lecture 3
August 29, 2016

1 Flux

Let's talk about flux. Illustrate with field lines. Line has length proportional to amplitude and direction of field. Or we can have continuous field lines and say that the density gives the amplitude. We can define a flux of $\mathbf{E}$ through a surface S.

If a charge is placed at the center of a spherical shell, the flux through the shell is q/ϵ_0 independent of the radius of the sphere. But suppose the charge is not at the center but a distance z from the center. Then we compute the flux as

$$\begin{aligned}\int \mathbf{E}\cdot d\mathbf{a} &= \frac{q}{4\pi\epsilon_0}\int \frac{R - z\cos\theta}{(R^2+z^2-2Rz\cos\theta)^{3/2}}R^2\sin\theta d\theta d\phi \\ &= \frac{2\pi q}{4\pi\epsilon_0}\int \frac{R-zx}{(R^2+z^2-2Rzx)^{3/2}}R^2dx\end{aligned}$$

where $|\mathbf{r}-\mathbf{r}'|$ the distance from the charge to the surface is $(R^2+z^2-2Rz\cos\theta)^{1/2}$ and the component of $\mathbf{r}-\mathbf{r}'$ along the radial direction is

$$\frac{R - z\cos\theta}{|\mathbf{r}-\mathbf{r}'|}$$

From an integral table we have that

$$\begin{aligned}\Phi &= \frac{q}{2\epsilon_0}\int \frac{R-zx}{(R^2+z^2-2Rzx)^{3/2}}R^2dx \\ &= \frac{q}{2\epsilon_0}\frac{Rx-z}{R^2\sqrt{R^2-2Rxz+z^2}}R^2|_{-1}^{1} \\ &= \frac{q}{2\epsilon_0}\left(\frac{R-z}{\pm(R-z)} - \frac{-R-z}{\pm(R+z)}\right) \\ &= \frac{q}{\epsilon_0}\end{aligned}$$

Suppose $z > R$. The argument of the square root is the distance from charge to surface when $\theta = 0, \pi$. It is always positive. So we take the $-$ sign for the first term and the $+$ sign for the second term. The sum is zero.

So we showed that the flux through a spherical shell due to a point charge anywhere inside the shell is q/ϵ_0 and zero if the charge is anywhere outside the shell. And we can distort the shell any way we like without changing the flux through the surface. It is after all the perpendicular component. The superposition principle tells us that we can put as many charges as we like inside (or outside) and we will find that the flux is the sum of the charges.

Then from the divergence theorem we know that

$$\int \mathbf{E} \cdot d\mathbf{a} = \int_v \nabla \cdot \mathbf{E} dv = \int_v \frac{\rho}{\epsilon_0} dv \rightarrow \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

This is equivalent to Coulomb's law. Depends only on the inverse square dependence and that the direction of the field is away from the charge.

2 Divergence

What about

$$\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

We could evaluate in cartesian coordinates. Or in spherical. In spherical coordinates

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r \sin\theta} \frac{\partial v_\phi}{\partial \phi}$$

and

$$\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = 0$$

everywhere except at the origin. Meanwhile we just determined that

$$\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot d\mathbf{a} = 4\pi$$

Then the divergence theorem says that

$$\int \nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} dv = 4\pi$$

If it is zero everywhere except the origin and the integral over any volume that includes the origin is the same, namely 4π then $\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$ since by definition

$$\int_v \delta^3(\hat{\boldsymbol{\mathscr{r}}})dv = 1$$

Armed with this information we can see that

$$\int \nabla \cdot \mathbf{E} = \int \nabla \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}\rho(\mathbf{r}')dv = \int 4\pi\delta^3(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}')dv = \frac{\rho}{\epsilon_0}$$

3 Gauss' Law

Field of long wire
Field of infinite plane
Field of spherical shell
Or uniformly charged sphere

4 Field in a slab of charge

Uniform charge in slab thickness 2d along y-axis. Outside the slab, the field is in y-direction and

$$EA = Ad\rho/\epsilon_0 \rightarrow E = \frac{\rho d}{\epsilon_0}$$

and inside

$$E = \frac{\rho y}{\epsilon_0}$$

5 Review

Coulomb's law

$$\Rightarrow \mathbf{E} = \frac{q}{4\pi\epsilon_0}\hat{\mathbf{r}}r^2$$

We showed that for charge at center of sphere

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{q}{\epsilon}$$

and then for q anywhere inside

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{q}{\epsilon}$$

(or outside)

$$\int \mathbf{E} \cdot d\mathbf{a} = 0$$

Then we argued for a surface enclosing the charge of any shape. And then by linear superposition

$$\int \mathbf{E} \cdot d\mathbf{a} = \int \frac{\rho}{\epsilon_0} dv$$

Then divergence theorem tells us that

$$\int \mathbf{E} \cdot d\mathbf{a} = \int \nabla \cdot \mathbf{E} dv \Rightarrow \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon}$$

6 Divergence

What about

$$\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

We could evaluate in cartesian coordinates. Or in spherical. In spherical coordinates

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

and

$$\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = 0$$

everywhere except at the origin. Meanwhile we just determined that

$$\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot d\mathbf{a} = 4\pi$$

Then the divergence theorem says that

$$\int \nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} dv = 4\pi$$

If it is zero everywhere except the origin and the integral over any volume that includes the origin is the same, namely 4π then $\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$ since by definition

$$\int_v \delta^3(\hat{\boldsymbol{\mathscr{r}}})dv = 1$$

Armed with this information we can see that

$$\int \nabla \cdot \mathbf{E} = \int \nabla \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}\rho(\mathbf{r}')dv = \int 4\pi\delta^3(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}')dv = \frac{\rho}{\epsilon_0}$$

Or in cartesian coordinates where

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$$

7 Field in a slab of charge

Uniform charge in slab thickness 2d along y-axis. Outside the slab, the field is in y-direction and

$$EA = Ad\rho/\epsilon_0 \rightarrow E = \frac{\rho d}{\epsilon_0}$$

and inside

$$E = \frac{\rho y}{\epsilon_0}$$

Lecture 4
August 31, 2016

8 Charge in the corner of a box

Box of side L with charge q in the corner. What is the flux through the opposite side. If we make the charge the center of a cube with sides of length 2L then use the fact that the total flux enclosed is q/ϵ_0, we see that the flux through the $L \times L$ section is $\frac{1}{6}\frac{1}{4}\frac{q}{\epsilon_0}$.

Or we could integrate

$$\begin{aligned}
\int \mathbf{E} \cdot d\mathbf{a} &= \frac{q}{4\pi\epsilon_0}\int_0^L\int_0^L \frac{L}{(x^2+z^2+L^2)^{3/2}}dxdz \\
&= \frac{q}{4\pi\epsilon_0}\int_0^L \frac{Lx}{(z^2+L^2)(x^2+z^2+L^2)^{1/2}}|_0^L dz \\
&= \frac{q}{4\pi\epsilon_0}\int_0^L \frac{L^2}{(z^2+L^2)(L^2+z^2+L^2)^{1/2}}dz \\
&= \frac{q}{4\pi\epsilon_0}\int_0^L \frac{L^2}{(z^2+L^2)(z^2+2L^2)^{1/2}}dz \\
&= \frac{q}{4\pi\epsilon_0}\frac{L^2\tan^{-1}\left(\frac{z}{\sqrt{2L^2+z^2}}\right)}{L^2}|_0^L \\
&= \frac{q}{4\pi\epsilon_0}\tan^{-1}\left(\frac{L}{\sqrt{2L^2+L^2}}\right) \\
&= \frac{q}{4\pi\epsilon_0}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\
&= \frac{q}{4\pi\epsilon_0}\frac{2\pi}{12} \\
&= \frac{q}{24\epsilon_0}
\end{aligned}$$

9 Curl of E

Consider the line integral of the E-field

$$\mathbf{E}\cdot d\mathbf{l} = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

And $d\mathbf{l} = \hat{\mathbf{r}}dr + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$ We find

$$\mathbf{E} \cdot d\mathbf{l} = -\frac{q}{4\pi\epsilon_0 r}|_a^b$$

The line integral is independent of the path and depends only on the end points and in fact only on the distance of the end points from the source. Then around a closed path

$$\mathbf{E} \cdot d\mathbf{l} = 0$$

Stokes theorem

$$\oint_P \mathbf{F} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{F} \cdot d\mathbf{a}$$

Therefore $\nabla \times \mathbf{E} = 0$ for static sources.

And in general for distributed source by linear superposition. This is a consequence of direction away from source $\hat{\mathbf{r}}$ and really nothing much to do with $\frac{1}{r^2}$. Could be any function of $f(\mathbf{r})\hat{\mathbf{r}}$. Well maybe not any function. Any function that depends only on $|\mathbf{r}|$. Like $1/r^3$ or $\frac{e^{-\mu r}}{r^2}$.

10 Potential

Since the electric field is a linear combination of contributions from charges q with contribution from each

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

and $\nabla \times \mathbf{E} = 0$ for a point charge then it must be generally true that $\nabla \times \mathbf{E} = 0$. Which means that the line integral is independent of the path. Namely since

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int \nabla \times \mathbf{E} \cdot d\mathbf{a}$$

the line integral around a closed loop is zero and therefore the lintegral between any two points in space is independent of the path.

$$\int_a^b \mathbf{E} \cdot d\mathbf{l}, \quad d\mathbf{l} = \hat{\mathbf{r}}dr + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$$

$$\mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}dr$$

and

$$\int_a^b \mathbf{E} \cdot d\mathbf{l} = -\frac{1}{4\pi\epsilon_0}\frac{q}{r}|_a^b = -\frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

If $r_a = r_b$ then

$$\oint \to \oint \mathbf{E} \cdot dbfl = 0$$

and from STokes theorem

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int_s (\nabla \times \mathbf{E}) \cdot d\mathbf{a}$$

Then by superpositon for any E field. Integral from a to b independent of path so

$$\int_O^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$$

defines a function $-V(\mathbf{r})$. Picture with collection of charge in region to loeft. points $\mathbf{a}$ and $\mathbf{b}$ and $\mathbf{r}$ to right. Integral from far away, where $\mathbf{E} = 0$ to those points gives potential.

Then calculus tells us that

$$V(b) - V(a) = \int_a^b \nabla V \cdot d\mathbf{l} \Rightarrow -\nabla \mathbf{V} = \mathbf{E}$$

Sphere with uniform surface charge. Use Gauss's law to get E. Then integrate to get $V(R)$ etc. Plot $V(r)$ and $E(r)$.

11 Poisson and Laplace

In a region where there is charge

$$-\nabla \cdot \nabla V = \frac{\rho}{\epsilon_0}$$

In empty space

$$-\nabla \cdot \nabla V = 0$$

12 Boundary conditions

$$\oint_S \mathbf{E} \cdot \hat{\mathbf{n}} da = \frac{Q_{end}}{\epsilon_0} = \frac{1}{\epsilon_0} \sigma A \Rightarrow E_\perp^{above} - E_\perp^{below} = \frac{\sigma}{\epsilon_0}$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \Rightarrow E_\parallel^{above} = E_\parallel^{below}$$

Summary

$$\mathbf{E}_{above} - \mathbf{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

Potential continuous at boundary. since $\int_a^b \mathbf{E} \cdot d\mathbf{l}$ across boundary $\to 0$. Gradient consistent with BC for E.

Lecture 5
September 2, 2016

13 Potential

Since the electric field is a linear combination of contributions from charges q with contribution from each

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

and $\nabla \times \mathbf{E} = 0$ for a point charge then it must be generally true that $\nabla \times \mathbf{E} = 0$. Which means that the line integral is independent of the path. Namely since

$$\oint \mathbf{E}{\cdot}d\mathbf{l} = \int \nabla \times \mathbf{E} \cdot d\mathbf{a}$$

the line integral around a closed loop is zero and therefore the lintegral between any two points in space is independent of the path.

$$\int_a^b \mathbf{E}{\cdot}d\mathbf{l}, \quad d\mathbf{l} = \hat{\mathbf{r}}dr + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$$

$$\mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}dr$$

and

$$\int_a^b \mathbf{E} \cdot d\mathbf{l} = -\frac{1}{4\pi\epsilon_0}\frac{q}{r}\,|_a^b = -\frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

If $r_a = r_b$ then

$$\oint \rightarrow \oint \mathbf{E}{\cdot}d\mathbf{l} = 0$$

and from Stokes theorem

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int_s (\nabla \times \mathbf{E}) \cdot d\mathbf{a}$$

Then by superpositon for any E field. Integral from a to b independent of path so

$$\int_O^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$$

defines a function $-V(\mathbf{r})$. Picture with collection of charge in region to loeft. points **a** and **b** and **r** to right. Integral from far away, where $\mathbf{E} = 0$ to those points gives potential.

Then calculus tells us that

$$V(b) - V(a) = \int_a^b \nabla V \cdot d\mathbf{l} \Rightarrow -\nabla \mathbf{V} = \mathbf{E}$$

Sphere with uniform surface charge. Use Gauss's law to get E. Then integrate to get $V(R)$ etc. Plot $V(r)$ and $E(r)$.

14 Poisson and Laplace

In a region where there is charge

$$-\nabla \cdot \nabla V = \frac{\rho}{\epsilon_0}$$

In empty space

$$-\nabla \cdot \nabla V = 0$$

15 Boundary conditions

$$\oint_S \mathbf{E} \cdot \hat{\mathbf{n}} da = \frac{Q_{end}}{\epsilon_0} = \frac{1}{\epsilon_0}\sigma A \Rightarrow E_\perp^{above} - E_\perp^{below} = \frac{\sigma}{\epsilon_0}$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \Rightarrow E_\parallel^{above} = E_\parallel^{below}$$

Summary

$$\mathbf{E}_{above} - \mathbf{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

Potential continuous at boundary. since $\int_a^b \mathbf{E} \cdot d\mathbf{l}$ across boundary $\rightarrow 0$. Gradient consistent with BC for E.

16 Work

Consider a charge q at $\mathbf{r_1}$. The efield of the charge is $\mathbf{E_1} = \frac{1}{4\pi\epsilon_0}\frac{q\mathbf{r}-\mathbf{r_1}}{|\mathbf{r}-\mathbf{r_1}|^3}$. The work done bringing another charge q' from far away to $\mathbf{r_2}$ a distance d from q_1 is

$$\int_\infty^d \mathbf{F} \cdot d\mathbf{l} = -q_2 \int_\infty^d \mathbf{E_1} \cdot d\mathbf{l} = q_2 V_1(\mathbf{r_2})$$

where $V(d)$ is the potential of the charge q at d. (The - sign is because the force we have to apply is opposite to the electric field. We might as well have put q_2 at the origin, figured out the potential and then brought q in from infinity to d. Same work. So the work is the charge times the potential of the other charge. And we can make it symmetric by writing

$$W = \frac{1}{2}(q_2 V_1 + q_1 V_2)$$

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i \left(\sum_{j \neq i}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{r_{ij}} \right) = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r_i}) \rightarrow \frac{1}{2}\int \rho V d\tau$$

$$W = \frac{1}{2}\int \rho V d\tau = \frac{\epsilon_0}{2}\left(\int_V E^2 d\tau + \oint_S V\mathbf{E} \cdot d\mathbf{a} \right) \rightarrow \frac{\epsilon_0}{2}\int_V E^2 d\tau$$