# Intermediate Electricity and Magnetism

September 12-16, 2016

Lecture 8
September 12, 2016

## 1 Laplace's Eqn.

Gauss's law says $\nabla \cdot \mathbf{E} = -\frac{\rho}{\epsilon_0}$ and $\mathbf{E} = -\nabla V$. Therefore

$$\nabla^2 V = \frac{\rho}{\epsilon_0}$$

and in a region of space with no charge

$$\nabla^2 V = 0.$$

In one dimension? Suppose twe have two infinite parallel plates in the y-z plane. With separation $d$ along $x$. The potential depends only on $x$. Let the potential on the plane at $x = 0$ is $V_0$ and the potential on the plane at $x = d$ is 0. Since the second derivative is zero every where in between, it must be that for all $x$ in between

$$V_0 > V(x) > 0$$

Because a zero second derivative indicates a maximum or minimum. In 2 and 3 dimensions things get a bit more complicated. We only know for sure that the sum of the second derivatives is zero. But we can say something like the value between the boundaries is some kind of average on the boundaries.

Cartesian coordinates

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Sperical coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Cylindrical coordinates

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

### 1.1 Average- 2 D

The average on a circle around a point is the same as the value at that point.

$$V(x,y) = V_{avg} = \frac{1}{2\pi R}\oint_{circle} V dl$$

So no local min or max. Say there is a max. Use it as the center of a circle. Average over the circle. The average would be less. So therefore no peaks allowed.

### 1.2 Average 3D

Pick any point in space $V(\mathbf{r})$. Use it as the center of a spherical boundary. Compute the average potential on the boundary. It will be equal to the value at the center.

$$V(\mathbf{r}) = V_{avg} = \frac{1}{4\pi R^2}\oint_{sphere} V da$$

Same reasonaing as before. No local max or min allowed.

Proof: Field of a point charge a distance $z$ from the center of a sphere. The goal is to average the potential of the point charge over the surface of the sphere and compare to the potential at the center of the sphere. The potential on the surface at $R, \theta, \phi$ is

$$V(R,\theta) = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{z^2 + R^2 - 2Rz\cos\theta}}$$

and

$$V da = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{z^2 + R^2 - 2Rz\cos\theta}} 2\pi R^2 sin\theta d\theta$$

$$V_{ave} = \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\int_0^\pi \frac{1}{\sqrt{z^2 + R^2 - 2Rz\cos\theta}} 2\pi R^2 sin\theta d\theta$$

$$V_{ave} = \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\int_{-1}^1 \frac{1}{\sqrt{z^2 + R^2 - 2Rzx}} 2\pi R^2 dx$$

$$V_{ave} = -\frac{2\pi R^2}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\sqrt{z^2 + R^2 - 2Rzx}\ |_{-1}^1$$

$$V_{ave} = -\frac{2\pi R^2}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\left((z-R)-(z+R)\right) = \frac{q}{4\pi\epsilon_0}\frac{1}{z}$$

which sure enough is the potential due to the charge at the center of the sphere. Thanks to supperposition, the same will be true for any charge distribution outside the sphere so it must be true in general.

## 2 Uniqueness 1

If we know the potential on the boundary of a region, then the potential everywhere inside the boundary is uniquely determined. Proof: Suppose there are two solutions $V_1$ and $V_2$. Then $V_3 = V_2 - V_1 = 0$ on the boundary. And $\nabla^2 V_3 = 0$. If it is zero on the boundaries and there are no local max or min it must be zero everywhere.Therefore $V_1 = V_2$. A result is that there is no way to electrostatically confine a charged particle.

We can always add some charge and thenthe potential in a volume $\mathcal{V}$ is uniquely determined if the charge density throughout the region is known and if the potential on the boundary is known.

## 3 Uniqueness 2 - conductors

A volume $\mathcal{V}$ is surrounded by conductors and there is some charge density $\rho$ between the conudctors and maybe on the conductors. The electric field is uniquely determined if the total charge on each conudctor is given. Note, the potential is not unique. We can always add some constant. We could connct each conductor to a battery at voltage $V_0$ and offset the whole deal.

Lecture 10
September 16, 2016

# 4 Separation of Variables - Spherical Coordinates

laplace's equation in spherical coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Assume azimuthal symmetry, multiply by $r^2$ and we get

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) = 0$$

Separate variables by writing $V(r,\theta) = R(r)\Theta(\theta)$. Substitute into differential equation above and divide by $R(r)\Theta(\theta)$ and we have

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{1}{\Theta \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta}{\partial \theta}\right) = 0$$

Solution requires that

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) = \text{constant} \equiv l(l+1)$$

$$\frac{1}{\Theta \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta}{\partial \theta}\right) = -l(l+1)$$

The general solution for

$$R(r) = Ar^l + \frac{B_l}{r^{l+1}}$$

and

$$\Theta(\theta) = P_l(\cos\theta)$$

# 5 Spherical Shell with charge distribution

Boundary conditions are $V(r \to \infty) = 0$, $V(0)$ is finite, and at the boundary of the surface, continuous. Inside solution

$$V_{in} = \sum_{l=0}^{\infty} A_l r^l P_l$$

and outside

$$V_{out} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l$$

$V_{out}$ is zero at $r = \infty$ and $V_{in}$ is finite at $r = 0$. Potential is continuous at a boundary so

$$V_{in} = \sum_{l=0}^{\infty} A_l R^l P_l = V_{out} = \sum_{l'=0}^{\infty} \frac{B_{l'}}{R^{l'+1}} P_{l'}$$

Multiply left and right by $P_{l''} \sin\theta d\theta$ and integrate from 0 to $\pi$ to get

$$\sum_{l=0}^{\infty} A_l R^l \frac{2\delta_{l,l''}}{2l+1} = \sum_{l'=0}^{\infty} \frac{B_{l'}}{R^{l'+1}} \frac{2\delta_{l',l''}}{2l'+1}$$

$$\to A_{l''} R^{l''} = \frac{B_{l''}}{R^{l''+1}}$$

and

$$B_l = A_l R^{2l+1}$$

There is another boundary condition at the surface of the sphere. We know from Gauss's law that

$$E_{\perp}^{above} - E_{\perp}^{below} = \frac{\sigma}{\epsilon_0}$$

and

$$E_{perp} = -\frac{\partial V}{\partial n}$$

At the surface of the sphere $\frac{\partial V}{\partial n} = \frac{\partial V}{\partial r}$ so

$$\frac{\partial V^{out}}{\partial r} - \frac{\partial V^{in}}{\partial r} = -\frac{\sigma(\theta)}{\epsilon_0}$$

Using everything we know so far we get that

$$\sum_{l=0}^{\infty} (2l+1) R^{l-1} P_l(\theta) = \frac{\sigma(\theta)}{\epsilon_0}$$

Multiply both sides by $P_{l'}(\cos\theta) \sin\theta d\theta$ and integrate and we get

$$A_l = \frac{1}{2R^{l-1}} \int_0^{\pi} \frac{\sigma(\theta)}{\epsilon_0} P_l(\cos\theta) \sin\theta d\theta$$

where we used $\int P_l P_{l'} \sin\theta d\theta = \frac{2}{2l+1}\delta_{l,l'}$ For the sphere where the top half has $\sigma = \sigma_0$ and the bottom $\sigma = -\sigma_0$

$$A_l = \frac{1}{2\epsilon_0 R^{l-1}}\left[\int_0^{\pi/2} P_l(\cos\theta) - \int_{\pi/2}^{\pi} P_l(\cos\theta)\right]\sin\theta d\theta$$

$$A_0 = 0$$

$$A_1 = \frac{1}{2\epsilon_0}$$

Even terms are zero. Next term is $A_3$. Note that $P_3 = (5\cos^3\theta - 3\cos\theta)/2$. Then

$$V_{out} \sim \frac{B_1}{r^2}\cos\theta = \frac{\sigma R^3}{2\epsilon_0}\frac{\cos\theta}{r^2}$$

# 6 Dipole moment

$$\mathbf{p} = \int \mathbf{r}'\rho(\mathbf{r}')d\tau'$$

Let's compute the dipole moment for that split sphere.

$$p = 2\left[2\pi R^2\int_0^{\pi/2}\sigma\sin\theta d\theta R\cos\theta\right] = 2\pi R^3\sigma$$

$$V_{dip} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} = \sigma\frac{R^3}{2\epsilon_0}\frac{\cos\theta}{r^2}$$

# 7 Conducting spherical shell in uniform E-field

The uniform E-field $\mathbf{E} = E_0\hat{\mathbf{z}}$ derives in spherical coordinates from a potential

$$V(r,\theta) = -E_0 z + C = -E_0 r\cos\theta + C$$

Set $C = 0$ so $V(0) = 0$. Check that by taking

$$\nabla \cdot V = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\phi}$$

At large $r$, $V(r,\theta) \to -E_0 r\cos\theta$. And at $r = 0$, $V(r) = 0$. We can write

$$V_{in} = \sim_{l=0}^{\infty} A_l^{in} r^l P_l$$

and

$$V_{out} = \sum \left( A_l^{out} r^l + \frac{B_l}{r^{l+1}} \right) P_l$$

The sphere is a conductor so the otential inside is everywhere the same, namely zero. All $A_l^{in} = 0$. To match the large $r$ boundary condition, $A_1 = -E_0$ and $A_l, l \neq 0 = 0$. The surface of the sphere is an equipotential so

$$\sum \left( A_l R^l + \frac{B_l}{r^{l+1}} \right) P_l = 0$$

Since the $P_l$ are orthogonal, the only way the sum is zero is if

$$A_l R^l = -\frac{B_l}{r^{l+1}}$$

for all $l$. There $A_1 = -\frac{B_1}{R^3}$ and all other $A_l$ and $B_l$ are zero. Therefore

$$V_{out} = -E_0(r - \frac{R^3}{r^2})P_1(\cos\theta) = -E_0(r - \frac{R^3}{r^2})\cos\theta$$

The induce charge

$$\sigma(\theta) = -\epsilon \left( \frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r} \right) |_{r=R} = \epsilon(3E_0)\cos\theta$$