Intermediate Electricity and Magnetism

September 19-23, 2016

Lecture 11
September 19, 2016

1 Separation of Variables - Spherical Coordinates

laplace's equation in spherical coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

Assume azimuthal symmetry, multiply by r^2 and we get

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) = 0$$

Separate variables by writing $V(r,\theta) = R(r)\Theta(\theta)$. Substitute into differential equation above and divide by $R(r)\Theta(\theta)$ and we have

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = 0$$

Solution requires that

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = \text{constant} \equiv l(l+1)$$

$$\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = -l(l+1)$$

The general solution for

$$R(r) = Ar^l + \frac{B_l}{r^{l+1}}$$

and

$$\Theta(\theta) = P_l(\cos\theta)$$

2 Spherical Shell with charge distribution

Boundary conditions are $V(r \to \infty) = 0$, $V(0)$ is finite, and at the boundary of the surface, continuous. Inside solution

$$V_{in} = \sum_{l=0}^{\infty} A_l r^l P_l$$

and outside

$$V_{out} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l$$

V_{out} is zero at $r = \infty$ and V_{in} is finite at $r = 0$. Potential is continuous at a boundary so

$$V_{in} = \sum_{l=0}^{\infty} A_l R^l P_l = V_{out} = \sum_{l'=0}^{\infty} \frac{B_{l'}}{R^{l'+1}} P_{l'}$$

Multiply left and right by $P_{l''} \sin\theta d\theta$ and integrate from 0 to π to get

$$\sum_{l=0}^{\infty} A_l R^l \frac{2\delta_{l,l''}}{2l+1} = \sum_{l'=0}^{\infty} \frac{B_{l'}}{R^{l'+1}} \frac{2\delta_{l',l''}}{2l'+1}$$

$$\to A_{l''} R^{l''} = \frac{B_{l''}}{R^{l''+1}}$$

and

$$B_l = A_l R^{2l+1}$$

There is another boundary condition at the surface of the sphere. We know from Gauss's law that

$$E_{\perp}^{above} - E_{\perp}^{below} = \frac{\sigma}{\epsilon_0}$$

and

$$E_{perp} = -\frac{\partial V}{\partial n}$$

At the surface of the sphere $\frac{\partial V}{\partial n} = \frac{\partial V}{\partial r}$ so

$$\frac{\partial V^{out}}{\partial r} - \frac{\partial V^{in}}{\partial r} = -\frac{\sigma(\theta)}{\epsilon_0}$$

Using everything we know so far we get that

$$\sum_{l=0}^{\infty} (2l+1) R^{l-1} P_l(\theta) = \frac{\sigma(\theta)}{\epsilon_0}$$

Multiply both sides by $P_{l'}(\cos\theta)\sin\theta d\theta$ and integrate and we get

$$A_l = \frac{1}{2R^{l-1}} \int_0^\pi \frac{\sigma(\theta)}{\epsilon_0} P_l(\cos\theta)\sin\theta d\theta$$

where we used $\int P_l P_{l'} \sin\theta d\theta = \frac{2}{2l+1}\delta_{l,l'}$ For the sphere where the top half has $\sigma = \sigma_0$ and the bottom $\sigma = -\sigma_0$

$$A_l = \frac{1}{2\epsilon_0 R^{l-1}} \left[\int_0^{\pi/2} P_l(\cos\theta) - \int_{\pi/2}^{\pi} P_l(\cos\theta) \right] \sin\theta d\theta$$

$$A_0 = 0$$

$$A_1 = \frac{1}{2\epsilon_0}$$

Even terms are zero. Next term is A_3. Note that $P_3 = (5\cos^3\theta - 3\cos\theta)/2$. Then

$$V_{out} \sim \frac{B_1}{r^2}\cos\theta = \frac{\sigma R^3}{2\epsilon_0}\frac{\cos\theta}{r^2}$$

3 Dipole moment

$$\mathbf{p} = \int \mathbf{r}' \rho(\mathbf{r}') d\tau'$$

Let's compute the dipole moment for that split sphere.

$$p = 2\left[2\pi R^2 \int_0^{\pi/2} \sigma \sin\theta d\theta R\cos\theta \right] = 2\pi R^3 \sigma$$

$$V_{dip} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} = \sigma \frac{R^3}{2\epsilon_0}\frac{\cos\theta}{r^2}$$

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4 Conducting spherical shell in uniform E-field

The uniform E-field $\mathbf{E} = E_0\hat{\mathbf{z}}$ derives in spherical coordinates from a potential

$$V(r,\theta) = -E_0 z + C = -E_0 r\cos\theta + C$$

Set $C = 0$ so $V(0) = 0$. Check that by taking

$$\nabla \cdot V = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$$

At large r, $V(r,\theta) \to -E_0 r\cos\theta$. And at $r = 0$, $V(r) = 0$. We can write

$$V_{in} = \sim_{l=0}^{\infty} A_l^{in} r^l P_l$$

and

$$V_{out} = \sum \left(A_l^{out} r^l + \frac{B_l}{r^{l+1}} \right) P_l$$

The sphere is a conductor so the otential inside is everywhere the same, namely zero. All $A_l^{in} = 0$. To match the large r boundary condition, $A_1 = -E_0$ and $A_l, l \neq 0 = 0$. The surface of the sphere is an equipotential so

$$\sum \left(A_l R^l + \frac{B_l}{r^{l+1}} \right) P_l = 0$$

Since the P_l are orthogonal, the only way the sum is zero is if

$$A_l R^l = -\frac{B_l}{r^{l+1}}$$

for all l. There $A_1 = -\frac{B_1}{R^3}$ and all other A_l and B_l are zero. Therefore

$$V_{out} = -E_0(r - \frac{R^3}{r^2})P_1(\cos\theta) = -E_0(r - \frac{R^3}{r^2})\cos\theta$$

The induce charge

$$\sigma(\theta) = -\epsilon\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r} \right)|_{r=R} = \epsilon(3E_0)\cos\theta$$

5 Polarization

We are looking for a way to describe fields in matter. At the very local level, the fields will change abruptly due to all te electrons and protons. So it is useful to think about the average efield over some volume that is large compared to an atom but small compared to the dimensions of the material.

The average electric field within a spherical volume is determined by the average dipole moment (the polarization) of the charge within the sphere. That is

$$\mathbf{E}_{ave} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3} = -\frac{1}{3\epsilon_0}\mathbf{P}$$

This is easy to show. Consider a spherical volume, radius R and single point charge q at $\mathbf{r} = \mathbf{a}$. Might as well put the charge on the z-axis. The average electric field within the sphere is in the z direction. The other directions cancel by symmetry. So we just need to integrate the z-component.

$$\langle E_z \rangle V = \frac{q}{4\pi\epsilon_0}\int \frac{r\cos\theta - a}{(a^2 + r^2 - 2ar\cos\theta)^{3/2}} r^2 dr \sin\theta d\theta d\phi$$

$$\langle E_z \rangle V = \frac{q}{4\pi\epsilon_0}\int \frac{rx - a}{(a^2 + r^2 - 2arx)^{3/2}} r^2 dr dx d\phi$$

$$\langle E_z \rangle V = \frac{2\pi q}{4\pi\epsilon_0 a^2}\int \frac{r - ax}{(a^2 + r^2 - 2arx)^{1/2}} \Big|_{-1}^{1} r^2 dr$$

$$\langle E_z \rangle V = \frac{q}{2\epsilon_0 a^2}\int \left(\frac{(\pm(r-a)}{(r-a)} - \frac{r+a}{r+a}\right) r^2 dr$$

$$\langle E_z \rangle V = -\frac{q}{2\epsilon_0 a^2} 2\int_0^a r^2 dr$$

$$\langle E_z \rangle V = -\frac{q}{2\epsilon_0 a^2}\frac{2}{3}a^3$$

$$\langle E_z \rangle = -\frac{q}{2\epsilon_0}\frac{2}{3}a\frac{1}{\frac{4}{3}\pi R^3}$$

$$\langle E_z \rangle = -\frac{p}{\epsilon_0}\frac{1}{4\pi R^3}$$

The average field is proportional to the dipole moment. The contribution to the average field from an arbitrary charge distribution will be superposition of the dipole moments of each of the charges. So that last equation becomes

$$\langle \mathbf{E} \rangle = -\frac{\mathbf{p}_{tot}}{\epsilon_0}\frac{1}{4\pi R^3}$$

And since the polarization is the average dipole moment over the volume,

$$\langle \mathbf{E} \rangle = -\frac{\mathbf{P}}{3\epsilon_0}$$

6 Field of a polarized object

The potential due to a dipole at $\mathbf{r}'$ is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

If it is a perfect dipole, then this is true for any $\mathbf{r}$. If the dipole moment

$$\mathbf{p} = \mathbf{P}(\mathbf{r}')d\tau'$$

Then

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d\tau'$$

The average field within a spherical volume is given by the polarization. And the field outside due to the polarization inside. (Of course if there is net charge in the volume, that will also contribute to the field outside as a monopole.) Next rearrange Using

$$\nabla'\frac{1}{|r-r'|} = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

Substituting into the expression for the potential

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \mathbf{P}(\mathbf{r}')\cdot\nabla'\frac{1}{\mathscr{r}}d\tau'$$

Integrate by parts

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\int \nabla'\cdot\left(\frac{\mathbf{P}}{\mathscr{r}}\right)d\tau' - \int \frac{1}{\mathscr{r}}\nabla\cdot\mathbf{P}d\tau'\right)$$

Then from the divergence theorem

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\int_S \left(\frac{\mathbf{P}\cdot d\mathbf{a}'}{\mathscr{r}}\right) - \int_V \frac{1}{\mathscr{r}}\nabla\cdot\mathbf{P}d\tau'\right)$$

Define $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$ and $\rho_b = -\nabla\cdot\mathbf{P}$ and the potential due to the polarized stuff becomes sum of contribution from surface charge and from volume charge

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\int_S \frac{\sigma_b}{\mathscr{r}}da' + \int_V \frac{\rho_b}{\mathscr{r}}d\tau'\right)$$

7 Uniformly polarized sphere

Uniform polarization $\mathbf{P}$ The bound surface charge

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P \cos\theta$$

where we suppose $\mathbf{P} = P\hat{\mathbf{z}}$. The bound volume charge density $\rho_b = 0$. We could calculate the potential by integrating the surface bound charge. But we already know that since the polarization is uniform, the field outside is equivalent to that of a dipole at $r = 0$, namel

$$V_{out} = \frac{4}{3}\pi R^3 P \frac{\cos\theta}{4\pi\epsilon_0 r^2}$$

and inside we just integrate to r

$$V_{in} = \frac{4}{3}\pi r^3 P \frac{\cos\theta}{4\pi\epsilon_0 r^2} = \frac{Pr\cos\theta}{3\epsilon_0}$$

The electric field inside

$$\mathbf{E} = -\nabla V = -\frac{P}{3\epsilon_0}\hat{\mathbf{z}}$$

8 Surface bound charge

Consider a cylinder of length d and cross section A and polarization P. The dipole moment of the cylinder is $p = PAd = qd$ where q is the charge at each end. Therefore $q = PA$ and $\sigma = P$. or $\mathbf{P} \cdot \hat{\mathbf{n}}$

9 Accumulation of Bound charge

If the polarization is uniform, there is no accumulation of bound charge. But if non-uniform, like a bunch of dipoles with negative end near $r = 0$ and positive end radially out, there is an accumulation of negative charge in the center. The positive is pushed through the surface of the volume. Then

$$\int_V \rho_b d\tau = -\oint_S \mathbf{P} \cdot d\mathbf{a} = -\int_V (\nabla \cdot \mathbf{P}) d\tau$$

Therefore

$$\rho_b = -\nabla \cdot \mathbf{P}$$