Physics 443
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Kaons

1 Quarks

According to the Standard Model there are 3 generations of quarks, with two quarks in each generation. Quarks have electric charge, color charge and weak charge. The electric charge couples the quarks by electromagnetic forces. The color charge couples them by the strong force, and the weak charge by the weak force. Within each generation one of the quarks has electric charge $\frac{2}{3}e$ and the other $-\frac{1}{3}e$ where e is the charge of an electron. The 6 quark $flavors$ are up(u), down(d), charm(c), strange(s), top(t), and bottom(b).

$$\begin{pmatrix} u \\ d' \end{pmatrix} \quad \begin{pmatrix} c \\ s' \end{pmatrix} \quad \begin{pmatrix} t \\ b' \end{pmatrix} \tag{1}$$

The u, c and t have charge $\frac{2}{3}e$ and d', s', and b' have charge $-\frac{1}{3}$. For each quark, there is an antiquark with opposite electric charge. Thanks the mechanism of confinement, we never see free quarks, but only quark bound states. There are bound states of 3 quarks, like the proton (uud) and neutron (udd) with charges 1 and 0 respectively.

The 3 quark states are called baryons. Every possible 3 quark combination of the light quarks (u,d, and s) has been observed. The masses of the quarks are very different as shown in the table. Except for the proton, the 3 quark bound states are all unstable. There are also quark anti-quark states like the $\pi^+(u\bar{d})$, $\pi^-(\bar{u}d)$, $K^0(\bar{s}d)$,$K^+(\bar{s}u)$, $\bar{K^0}(s\bar{d})$, etc. with charges 1, -1, 0, 1, 0. The quark anti-quark states are called mesons. Every quark antiquark pair will appear as a meson.

Quark masses (MeV/c^2)	
u	10
d	10
s	80
c	1.55
b	4700
t	175000

We usually produce mesons in strong or electromagnetic interactions. They decay by weak interactions. Strong and electromagnetic interactions conserve quark flavor. The number of quarks - antiquarks of each flavor in the initial state is the same as the number in the final state. That is $N_q - N_{\bar{q}}$

is conserved. So a u quark cannot turn into a d in a strong interaction. On the other hand, weak interactions couple the charge $\frac{2}{3}$ quark with the charge $-\frac{1}{3}$ quark in each generation and the weak interaction is mediated by the charged vector boson call the W. The W has electric charge ± 1. The interactions are represented by Feynmann diagrams. Some of the allowed weak interaction vertices are shown in Figure 1.

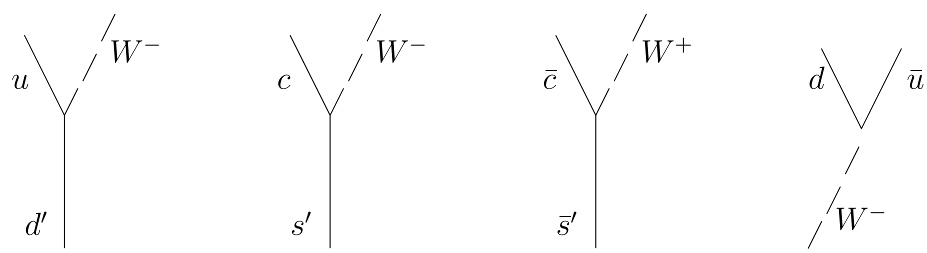

Figure 1: Weak interaction charged current vertices

Time runs up in the diagrams. Note that charge is conserved at each vertex. That is, the sum of the electric charges of the quarks and W's coming into the vertex is the same as the sum of electric charges leaving the vertex. The bar indicates the antiparticle so $\bar{q}$ is the anti q. Since we never see free quarks, the vertices indicated above are just pieces of more complicated interactions that involves mesons and baryons.

An important subtlety is that the quark flavors that we identify for participation in weak interactions according to 1 are not quite the same as the quarks states that couple by strong interaction. In particular the weak states s' and d' are linear combinations of the strong states s and d. Namely

$$\begin{aligned} s' &= s\cos\theta_c - d\sin\theta_c \\ d' &= d\cos\theta_c + s\sin\theta_c \end{aligned}$$

The mixing angle θ_c, the so-called Cabbibo angle has a value of about $13°$. So the s' and d' lines in Figure 1 are more precisely $s\cos\theta_c - d\sin\theta_c$ and $d\cos\theta_c + s\sin\theta_c$ lines.

The K^0 meson, the bound state of $\bar{s}$ and d quarks, is produced in the strong interaction process

$$\pi^-(d\bar{u}) + p(uud) \to \Lambda^0(sud) + K^0(\bar{s}d)$$

The quark content of the mesons and baryons is indicated in parenthesis. Note that flavor is conserved in this process.

We are interested in the fate of the K^0. If left alone, it is stable with respect to both strong and electromagnetic interactions since in both flavor is conserved. To understand what becomes of the K^0 we need to consider weak processes.

The fundamental weak interacton vertices indicated in Figure 1 can be combined in many different ways. The processes relevant to our discussion are shown in figure 2 and correspond to

$$K^0 \rightarrow \pi^+\pi^-, \quad K^0 \rightarrow \bar{K}^0 \quad \text{and} \quad K^0 \rightarrow K^0$$

Sometimes the K^0 decays to $\pi^+\pi^-$. Sometimes through an internal exchange of virtual W's, it changes into its own antiparticle. And then sometimes, by an equivalent virtual exhange it emerges back in its inital state.

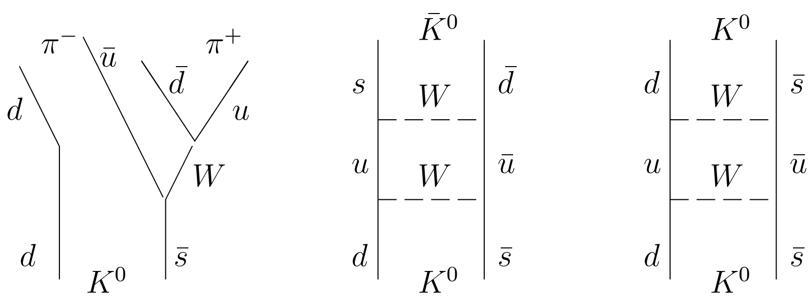

Figure 2: a) $K^0 \rightarrow \pi^+\pi^-$, b) $K^0 \rightarrow \bar{K}^0$, c) $K^0 \rightarrow K^0$

2 Two state system

Evidently the K^0 has some probability to change into its own antiparticle, the $\bar{K}^0$.

$$K^0 \leftrightarrow \bar{K}^0$$

A particle that begins life as a K^0 may appear at a later time as a $\bar{K}^0$. We will consider the K^0 and $\bar{K}^0$ as two distinct states of a single system. And write the state vector as a linear combination of the two base states.

$$|\,\psi\rangle = a|\,K^0\rangle + b|\,\bar{K}^0\rangle$$

The states $| K^0\rangle$ and $| \bar{K}^0\rangle$ are not stationary. We want to understand how the system changes in time.

3 Hamiltonian of the Two State System

In the K^0, $\bar{K}^0$ basis, the Hamiltonian matrix

$$H = \begin{pmatrix} \langle K^0 \mid H \mid K^0\rangle & \left\langle K^0 \mid H \mid \bar{K}^0\right\rangle \\ \left\langle \bar{K}^0 \mid H \mid K^0\right\rangle & \left\langle \bar{K}^0 \mid H \mid \bar{K}^0\right\rangle \end{pmatrix}$$

Let's separate the hamiltonian into the weak interaction piece W and then everything else H_0. Then $H = H_0 + W$. The weak interaction part includes all of the dynamics represented by the diagrams. H_0 is the hamiltonian of the freely propagating K^0. Then

$$\left\langle \bar{K}^0 \mid H \mid K^0\right\rangle = \left\langle \bar{K}^0 \mid H_0 + W \mid K^0\right\rangle = \left\langle \bar{K}^0 \mid W \mid K^0\right\rangle \tag{2}$$

The off diagonal terms correspond to the middle diagram in Figure 2. The inner product $\left\langle \bar{K}^0 \mid W \mid K^0\right\rangle$ is equal to some complex number A. The amplitude to go from K^0 to $\bar{K}^0$ must be the same as the amplitude for the reverse so

$$\left\langle K^0 \mid W \mid \bar{K}^0\right\rangle = \left\langle \bar{K}^0 \mid W \mid K^0\right\rangle = A$$

We also suppose that the amplitude for diagram c in Figure 2 is the same as for diagram b. So that

$$\left\langle \bar{K}^0 \mid W \mid \bar{K}^0\right\rangle = \left\langle K^0 \mid W \mid K^0\right\rangle = A$$

Finally if there is no weak interaction at all we have

$$\left\langle \bar{K}^0 \mid H_0 \mid \bar{K}^0\right\rangle = \left\langle K^0 \mid H_0 \mid K^0\right\rangle = E_0$$

Then

$$H = \begin{pmatrix} E_0 + A & A \\ A & E_0 + A \end{pmatrix} \tag{3}$$

Now

$$H| \psi\rangle = i\hbar\frac{\partial}{\partial t}| \psi\rangle$$

and if

$$| \psi(t)\rangle = a| K^0\rangle + b| \bar{K}^0\rangle$$

then

$$H\left(a|\ K^0\rangle + b|\ \bar{K}^0\rangle\right) = i\hbar\left(\frac{\partial a}{\partial t}|\ K^0\rangle + \frac{\partial b}{\partial t}|\ \bar{K}^0\rangle\right) \tag{4}$$

If we take the inner product of the terms in Equation 4 first with $\langle K^0 \mid$ and then with $\langle \bar{K}^0 \mid$ we get

$$\begin{aligned} a\left\langle K^0 \mid H \mid K^0\right\rangle + b\left\langle K^0 \mid H \mid \bar{K}^0\right\rangle &= i\hbar\frac{\partial a}{\partial t} \\ a\left\langle \bar{K}^0 \mid H \mid K^0\right\rangle + b\left\langle \bar{K}^0 \mid H \mid \bar{K}^0\right\rangle &= i\hbar\frac{\partial b}{\partial t} \end{aligned}$$

where we have used $\langle K^0 \mid K^0\rangle = \left\langle \bar{K}^0 \mid \bar{K}^0\right\rangle = 1$ and $\left\langle \bar{K}^0 \mid K^0\right\rangle = \left\langle K^0 \mid \bar{K}^0\right\rangle = 0$. In matrix form

$$\begin{aligned} \begin{pmatrix} \langle K^0 \mid H \mid K^0\rangle & \left\langle K^0 \mid H \mid \bar{K}^0\right\rangle \\ \left\langle \bar{K}^0 \mid H \mid K^0\right\rangle & \left\langle \bar{K}^0 \mid H \mid \bar{K}^0\right\rangle \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} &= \\ \begin{pmatrix} E_0 + A & A \\ A & E_0 + A \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} &= i\hbar\begin{pmatrix} \frac{\partial a}{\partial t} \\ \frac{\partial b}{\partial t} \end{pmatrix} \end{aligned}$$

Now

$$\mid \psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The base states $\mid K^0\rangle$ and $\mid \bar{K}^0\rangle$ are represented by the normalized state vectors

$$\mid K^0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mid \bar{K}^0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

4 Diagonalize the hamiltonian

The coupled differential equations are easiest to solve if we diagonalize the hamiltonian and then transform to the basis of energy eigenstates.

$$\begin{vmatrix} E_0 + A - \lambda & A \\ A & E_0 + A - \lambda \end{vmatrix} = 0$$

yields the eigenvalues. We find

$$\lambda_1 = E_0$$

$$\lambda_2 \quad = \quad E_0 + 2A$$

The energies of the stationary states are $E_1 = \lambda_1 = E_0$ and $E_2 = \lambda_2 = E_0 + 2A$. The normalized eigenvectors are

$$\begin{aligned}\vec{v}_1 &= \sqrt{\frac{1}{2}}\begin{pmatrix}1\\-1\end{pmatrix}\\ \vec{v}_2 &= \sqrt{\frac{1}{2}}\begin{pmatrix}1\\1\end{pmatrix}\end{aligned}$$

where

$$H\vec{v}_i = E_i\vec{v}_i \tag{5}$$

and $\langle v_i \mid v_j\rangle = \delta_{ij}$. We label the eigenstates $\mid K_1\rangle$ and $\mid K_2\rangle$.

5 Transformation to the diagonal basis

We can do a similarity transform to find the representation of the Hamiltonian and the state vectors in the basis in which the Hamiltonian is diagonal. It must be that

$$S^{-1}HS = H' \tag{6}$$

where

$$H = \begin{pmatrix}E_0 + A & A\\ A & E_0 + A\end{pmatrix} \quad \text{and} \quad H' = \begin{pmatrix}E_1 & 0\\ 0 & E_2\end{pmatrix}$$

The columns of the matrix S are the eigenvectors of H. Then

$$S = \sqrt{\frac{1}{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$$

and

$$\sqrt{\frac{1}{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}E_0 + A & A\\ A & E_0 + A\end{pmatrix}\sqrt{\frac{1}{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}E_1 & 0\\ 0 & E_2\end{pmatrix}$$

We can rearrange Equation 6 and rewrite Equation 5

$$\left(SH'S^{-1}\right)\vec{v}_i = E_i\vec{v}$$

Then

$$H'\left(S^{-1}\vec{v}_i\right) = E_i\left(S^{-1}\vec{v}_i\right)$$

and

$$H'\vec{v}_i' = E_i\vec{v}_i'$$

where $\vec{v}_i' = S^{-1}\vec{v}_i$. We compute

$$\vec{v}_i' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{v}_i' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\vec{v}_i'$ are the eigenvectors in the diagonal basis and correspond to stationary states

$$\begin{aligned} \mid K_1\rangle &= \vec{v}_1' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \mid K_2\rangle &= \vec{v}_2' = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \tag{7}$$

Finally we can transform the states $\mid K^0\rangle$ and $\mid \bar{K}^0\rangle$ to the diagonal basis.

$$\begin{aligned} \mid K^0\rangle \to S^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \sqrt{\frac{1}{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \mid \bar{K}^0\rangle \to S^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \sqrt{\frac{1}{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned} \tag{8}$$

Now the Schrodinger equation is easy to solve. In the diagnonal basis Equation 5 becomes

$$\begin{aligned} \begin{pmatrix} \langle K_1 \mid H \mid K_1\rangle & \langle K_1 \mid H \mid K_2\rangle \\ \langle K_2 \mid H \mid K_1\rangle & \langle K_2 \mid H \mid K_2\rangle \end{pmatrix}\begin{pmatrix} a' \\ b' \end{pmatrix} &= \\ \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}\begin{pmatrix} a' \\ b' \end{pmatrix} &= i\hbar\begin{pmatrix} \frac{\partial a'}{\partial t} \\ \frac{\partial b'}{\partial t} \end{pmatrix} \end{aligned}$$

And by inspection we see that

$$\begin{aligned} a'(t) &= a'(0)e^{-iE_1t/\hbar} \\ b'(t) &= b'(0)e^{-iE_2t/\hbar} \end{aligned}$$

Therefore

$$\mid \psi(t)\rangle = a'(0)e^{-iE_1t/\hbar}\mid K_1\rangle + b'(0)e^{-iE_2t/\hbar}\mid K_2\rangle$$

6 Time evolution

K^0's are produced in collisions of a π^- beam with a proton target.

$$\pi^- + p \rightarrow \Lambda_0 + K^0.$$

At $t = 0$

$$\mid \psi(t=0)\rangle = \mid K^0\rangle$$

In the diagonal basis

$$\psi(t=0) = \begin{pmatrix} a'(0) \\ b'(0) \end{pmatrix} = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

At time t,

$$\psi(t) = \begin{pmatrix} a'(t) \\ b'(t) \end{pmatrix} = \sqrt{\frac{1}{2}} \begin{pmatrix} e^{-iE_1 t/\hbar} \\ e^{-iE_2 t/\hbar} \end{pmatrix}$$

or in Dirac notation

$$\mid \psi(t)\rangle = \sqrt{\frac{1}{2}} \left(e^{-iE_1 t/\hbar} \mid K_1\rangle + e^{-iE_2 t/\hbar} \mid K_2\rangle \right)$$

The probability to find the particle in the state K_1 at time t is

$$|\langle K_1 \mid \psi(t)\rangle|^2 = \left| \sqrt{\frac{1}{2}} \left(e^{-iE_1 t/\hbar} \langle K_1 \mid K_1\rangle + e^{-iE_2 t/\hbar} \langle K_1 \mid K_2\rangle \right) \right|^2 = \left| \sqrt{\frac{1}{2}} e^{-iE_0 t/\hbar} \right|^2 = \frac{1}{2}$$

Note that we have used the fact that $E_1 = E_0$ is real so that $|e^{-iE_0 t/\hbar}|^2 = 1$. The probability to find the particle in the state K_2 at time t is

$$|\langle K_2 \mid \psi(t)\rangle|^2 = \left| \sqrt{\frac{1}{2}} \left(e^{-iE_1 t/\hbar} \langle K_2 \mid K_1\rangle + e^{-iE_2 t/\hbar} \langle K_2 \mid K_2\rangle \right) \right|^2 = \left| \sqrt{\frac{1}{2}} e^{-i(E_0 + 2A) t/\hbar} \right|^2$$

But now we have to be a bit more careful.

7 K^0 decay

We have been treating the K-meson as though it were a two state system, a K^0, $\bar{K}^0$ or in the diagonal basis as a K_1,K_2. Of course that is not a complete description. Sometimes the K-meson decays into $\pi^+\pi^-$ and then it is not in any of the base states. A precise treatment would require that we include all possible final states of the meson. We make a phenomonologial approximation by allowing the matrix element A to be complex.

We have that

$$|\langle K_2 \mid \psi(t)\rangle|^2 = \left|\sqrt{\frac{1}{2}}e^{-i(E_0+2A)t/\hbar}\right|^2$$

We write $A = \alpha - i\beta$. Then

$$|\langle K_2 \mid \psi(t)\rangle|^2 = \frac{1}{2}e^{-(4\beta)t/\hbar} \tag{9}$$

The K_2 component of the beam decays into $\pi^+\pi^-$ with time constant $1/\tau = -4\beta$. The probability for K_1 is independent of time. Of course we have no idea what β is. We will have to measure it.

What about the probability that we will find K^0 and $\bar{K}^0$ at later times?

$$\begin{aligned}
|\left\langle K^0 \mid \psi(t)\right\rangle|^2 &= \left|\sqrt{\frac{1}{2}}\left(e^{-iE_1t/\hbar}\left\langle K^0 \mid K_1\right\rangle + e^{-iE_2t/\hbar}\left\langle K^0 \mid K_2\right\rangle\right)\right|^2 \\
&= \left|\frac{1}{2}\left(e^{-iE_1t/\hbar} + e^{-iE_2t/\hbar}\right)\right|^2 \\
&= \frac{1}{4}\left(1 + e^{-4\beta t/\hbar} + 2e^{-2\beta t/\hbar}\cos(2\alpha t/\hbar)\right)
\end{aligned}$$

and

$$\begin{aligned}
|\left\langle \bar{K}^0 \mid \psi(t)\right\rangle|^2 &= |\sqrt{\frac{1}{2}}\left(e^{-iE_1t/\hbar}\left\langle \bar{K}^0 \mid K_1\right\rangle - e^{-iE_2t/\hbar}\left\langle \bar{K}^0 \mid K_2\right\rangle\right)|^2 \\
&= \left|\frac{1}{2}\left(e^{-iE_1t/\hbar} - e^{-iE_2t/\hbar}\right)\right|^2 \\
&= \frac{1}{4}\left(1 + e^{-4\beta t/\hbar} - 2e^{-2\beta t/\hbar}\cos(2\alpha t/\hbar)\right)
\end{aligned}$$

We have used the fact that $\langle K^0 \mid K_1\rangle = \langle K^0 \mid K_2\rangle = \left\langle \bar{K}^0 \mid K_1\right\rangle = -\left\langle \bar{K}^0 \mid K_2\right\rangle = \sqrt{\frac{1}{2}}$ which you can compute using Equations 7 and 8.

At $t = 0$ the probability is 1 that we find K^0. That's good because that was how we started. The probability oscillates with frequency $\alpha/\hbar$ and

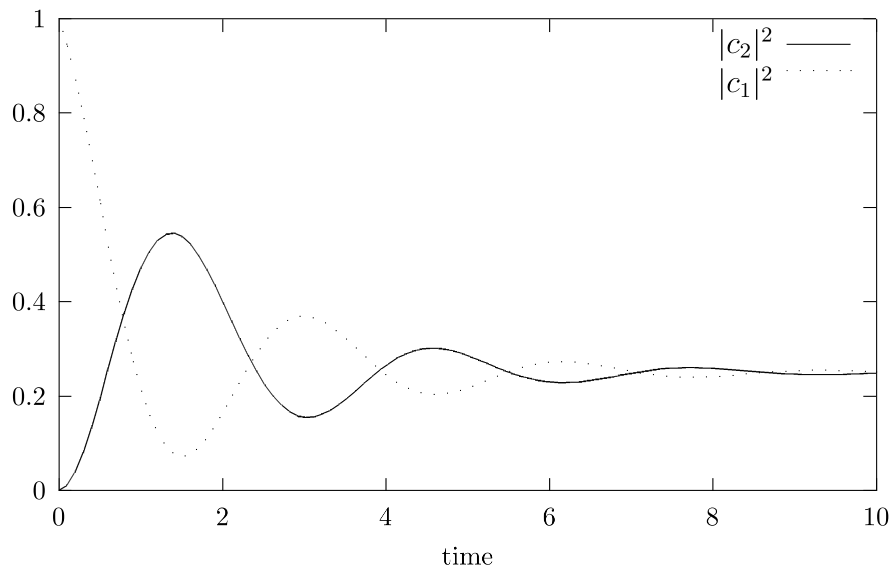

Figure 3: $|c_1|^2$ is the probability for K^0 and $|c_2|^2$ the probability for $\bar{K}^0$. The time units are arbitrary.

decays with time constant $1/\tau \sim \beta/\hbar$. As $t \to \infty$, the probability for K^0 approaches 1/4. Meanwhile, at $t = 0$, there is no chance that we will see $\bar{K}^0$ but the probability grows and oscillates and again at long times approaches 1/4. After a long time, half of the Kaons are left and half of those are K^0 and half are $\bar{K}^0$.

8 The experiment

This is what we measure. A beam of pure K^0 is produced in the $\pi^- + p$ collisions. Pure K^0 means $\frac{1}{2}$ K_1 and $\frac{1}{2}$ K_2. The K_2 decay into $\pi^+\pi^-$. (The target is typically liquid hydrogen.) By measuring the trajectories of the $\pi^+\pi^-$ we can determine the position of the K_2 when it decayed. We count the number of $\pi^+\pi^-$ decays versus distance from the proton target. The number will drop off exponentially so that we can determine β in Equation 9. Beyond a distance from the proton target corresponding to several decay lengths we will no longer see $\pi^+\pi^-$. All of the K_2 are gone and the beam is pure K_1.

We would also like to measure the K^0 and $\bar{K}^0$ content of the beam. Because the constituent quarks interact differently with ordinary matter, it is not difficult to distinguish the two kaon states. In strong interactions, quark flavor is conserved so we can have

$$K^0(\bar{s}d) + p(uud) \rightarrow K^+(\bar{s}u) + n(udd)$$

or

$$\bar{K}^0(s\bar{d}) + p(uud) \rightarrow \pi^+(\bar{d}u) + \Lambda^0(sud)$$

but a K^0 will never produce a Λ^0 and a π^+ and a $\bar{K}^0$ will never yield a K^+ and a neutron. We place a tungsten target in the path of the kaon beam (tungsten is very dense and has lots of protons and neutrons) and the number of neutrons that come out will be proportional to the number of K^0 that went in, and the number of π^+ will be proportional to the number of $\bar{K}^0$. In this way we can measure the relative number of K^0 and $\bar{K}^0$ in the beam at any point along its trajectory by simply relocating the tungsten. The dependence is as in the plot and the period of oscillation gives us α.

9 Regeneration

Because the K^0 and $\bar{K}^0$ interact differently with matter, the probability of absorption in passage through some material will be different as well. Suppose that we are far from the proton production target and all of the K_2 have decayed. Then

$$\mid \psi\rangle = \mid K_1\rangle = \sqrt{\frac{1}{2}}(\mid K^0\rangle - \mid \bar{K}^0\rangle)$$

If this beam is directed into an absorber, then when it emerges, it will be in some state

$$\mid \psi\rangle = \sqrt{\frac{1}{2}}(f\mid K^0\rangle - \bar{f}\mid \bar{K}^0\rangle)$$

where f and $\bar{f}$ represent the transmission amplitudes for K^0 and $\bar{K}^0$ respectively. Then emerging from the absorber

$$\mid \psi\rangle = \sqrt{\frac{1}{2}}(f+\bar{f})(\mid K^0\rangle - \mid \bar{K}^0\rangle) + \sqrt{\frac{1}{2}}(f-\bar{f})(\mid K^0\rangle + \mid \bar{K}^0\rangle) = (f+\bar{f})\mid K_1\rangle + (f-\bar{f})\mid K_2\rangle$$

The K_2 content of the beam has been regenerated (as long as f and $\bar{f}$ are different) and we will again begin to see $\pi^+\pi^-$ produced along the kaon beam trajectory.

10 $K_1 \to \pi^+\pi^-\pi^0$

For the record, the K_1 is not really stable. It can decay to 3π's but the rate for $K_1 \to \pi^+\pi^-\pi^0$ is about 600 times slower than the rate for $K_2 \to \pi^+\pi^-$ so our assumption that it lives forever is not a bad approximation. The lifetime of the K_2 is about $\tau = 0.9 \times 10^{-10}$ seconds. The lifetime of K_1 is $\tau = 0.5 \times 10^{-7}$ seconds.

11 References

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