Sample P443 Prelim Questions
March 6, 2008

1. **Harmonic Oscillator**

   A harmonic oscillator is in a state such that the measurement of the energy would yield either $\frac{1}{2}\hbar\omega$ or $\frac{3}{2}\hbar\omega$ with equal probability.

   (a) What is the expectation value of the energy?

   (b) What is the largest possible value of $\langle x \rangle$ in such a state?

   (c) If it assumes this maximal value at $t = 0$, what is $\Psi(x, t)$? (Give the answer in terms of $\psi_0(x)$ and $\psi_1(x)$.)

   (d) What is the smallest possible value of $\langle x \rangle$ in such a state?

   The raising and lowering operators are

   $$a_\pm = \frac{\hat{p}}{\sqrt{2m}} \pm i\sqrt{\frac{m}{2}}\omega\hat{x}$$

   and the hamiltonian is

   $$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = a_+a_- + \frac{1}{2}\hbar\omega$$

   And

   $$\begin{aligned} a_+|\, n\rangle &= i\sqrt{(n+1)\hbar\omega}|\, n+1\rangle \\ a_-|\, n\rangle &= -i\sqrt{n\hbar\omega}|\, n-1\rangle \end{aligned}$$

   $$H|\, n\rangle = E_n|\, n\rangle = (n + \frac{1}{2})\hbar\omega|\, n\rangle$$

2. **Reflection**

   A particle of mass m and kinetic energy $E > 0$ is traveling in the positive x-direction when at $x = 0$ there is a delta-function potential

   $$V(x) = \alpha\delta(x)$$

   (a) What is the probability that it will reflect back?

(b) What is the probability that it will be transmitted.

3. **Spin 1/2**

   Consider a spin 1/2 particle with magnetic moment $\vec{\mu} = \frac{e}{m}\vec{S}$ in a magnetic field $\vec{B} = B_0\hat{z}$. The hamiltonian

   $$H = -\vec{\mu} \cdot \vec{B} = \frac{e}{m}\vec{S} \cdot \vec{B}$$

   where $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$.

   Find the eigenvectors (wave functions) and the eigenvalues (energy levels) of the hamiltonian.

4. **Free Particle**

   Suppose a free particle, which is initially localized in the range $-a < x < a$ is released at $t = 0$.

   $$\Psi(x, 0) = \begin{cases} A, & \text{if } -a < x < a, \\ 0, & \text{otherwise} \end{cases}$$

   where A and a are real constants.

   (a) Determine A, by normalizing Ψ.

   (b) Determine $\phi(k)$.

   (c) Determine $\Psi(x, t)$. You can leave the answer as an integral.

5. **Square well**

   A particle in the inifinite square well has as its initial wave function an even mixture of the first two stationary states:

   $$\Psi(x, 0) = A[\psi_1(x) + \psi_2(x)]$$

   where

   $$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a}$$

   (a) Normalize $\Psi(x, 0)$.

   (b) Find $\Psi(x, t)$ and $|\Psi(x, t)|^2$

(c) Compute $\langle x \rangle$. (You don't have to evaluate the integrals.) What is the frequency of oscillation?

(d) Find the expectation value of H.

6. **Ehrenfest's theorem**

Use Schrodinger's equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

to calculate

$$\frac{d\langle p \rangle}{dt}$$

and show that the expectation values obey $F = ma$.

7. **Half Harmonic Oscillator**

Find the allowed energies of the half-harmonic oscillator.

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2x^2, & \text{for}(x > 0), \\ \\ \infty, & \text{for}(x < 0) \end{cases}$$

8. **Reflection**

A particle of mass m and kinetic energy $E > 0$ is traveling in the positive x-direction when at $x = 0$ there is an abrupt potential drop. That is,

$$V(x) = \begin{cases} 0, & \text{for}(x < 0), \\ \\ -V_0, & \text{for}(x > 0) \end{cases}$$

What is the probability that it will reflect back?

9. **Eigenvalues and eigenvectors**

Consider the 2X2 matrix

$$T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Find the eigenvalues and the normalized eigenvectors. Construct the matrix S that diagonalizes T so that

$$STS^-1 = T'$$

and T' is diagonal.

10. **Spin 1**

Consider a system with total angular momentum 3/2 and with basis vectors

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \chi_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \chi_{-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

Derive matrix representations for $J_\pm, J_z, J^2$ and J_y.

11. **Neutrinos**

Assume that there are two distinct neutrino states $|\ \nu_1\rangle$ and $|\ \nu_2\rangle$ with definite and distinct masses m_1 and m_2. So for example, if at $t = 0$, $|\ \psi(t=0)\rangle = |\ \nu_1\rangle$, then $|\ \psi(t)\rangle = |\ \nu_1\rangle e^{-i\frac{E_1}{\hbar}t}$. The mass eigenstates are linear combinations of the weak interaction eignenstates $|\ \nu_e\rangle$ and $|\ \nu_\mu\rangle$ and

$$\begin{aligned} |\ \nu_1\rangle &= \cos\theta |\ \nu_e\rangle + \sin\theta |\ \nu_\mu\rangle \\ |\ \nu_2\rangle &= -\sin\theta |\ \nu_e\rangle + \cos\theta |\ \nu_\mu\rangle \end{aligned}$$

where θ is a mixing angle.

(a) Suppose that the neutrino is born in the state $|\ \nu_e\rangle$ at time $t = 0$ so that $|\ \psi(0)\rangle = |\ \nu_e\rangle = \cos\theta |\ \nu_1\rangle - \sin\theta |\ \nu_2\rangle$. And suppose that the neutrino has a definite linear momentum p and that $p^2c^2 \gg m^2c^4$ so that

$$E_i = \sqrt{p^2c^2 + m_i^2c^4} \approx pc(1 + m^2c^2/(2p^2); \quad i = 1, 2$$

What is the neutrino state $|\ \psi(t)\rangle$ at $t > 0$? Give your answer in the $|\ \nu_1\rangle, |\ \nu_2\rangle$ basis.

(b) What is the probability that the neutrino born as $|\ \nu_e\rangle$ has evolved to $|\ \nu_\mu\rangle$ at time t? Give your answer in terms of $\Delta m^2 = m_1^2 - m_2^2$

12. **WKB**

Consider the spherically symmetric potential $V(r) = kr$. The radial form of Scrhodinger's equation, with $l = 0$ is

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V(r)u = Eu$$

(a) Write an expression for the turning point r_0 in terms of E_n and k.

(b) Use the WKB approximation to estimate the allowed energies of a particle in this potential with zero angular momentum.

- $$\begin{aligned} H\Psi &= i\hbar\frac{\partial\Psi}{\partial t} \\ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) &= i\hbar\frac{\partial}{\partial t}\Psi(x,t) \end{aligned}$$

- WKB

$$\int_{x_1}^{x_2} p(x)dx = \left(n-\frac{1}{2}\right)\pi\hbar \quad \text{(no infinite vertical walls)}$$

$$\int_{x_1}^{x_2} p(x)dx = \left(n-\frac{1}{4}\right)\pi\hbar \quad \text{(1 infinite vertical wall)}$$

$$\int_{x_1}^{x_2} p(x)dx = n\pi\hbar \quad \text{(2 infinite vertical walls)}$$

$$\psi(x) = \frac{A}{\sqrt{p}}\exp(\frac{i}{\hbar}\int^x p(x')dx') + \frac{B}{\sqrt{p}}\exp(-\frac{i}{\hbar}\int^x p(x')dx')$$

$$\psi(x) = \frac{C}{\sqrt{|p|}}\exp(\frac{1}{\hbar}\int^x |p(x')|dx') + \frac{D}{\sqrt{|p|}}\exp(-\frac{1}{\hbar}\int^x |p(x')|dx')$$

- One dimensional harmonic oscillator

$$\begin{aligned} H &= \frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2 \\ a_\pm &= \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p}+m\omega\hat{x}) \\ a_+\psi_n &= \sqrt{(n+1)}\psi_{n+1} \\ a_-\psi_n &= \sqrt{n}\psi_{n-1} \\ [a_-,a_+] &= 1 \\ H &= \hbar\omega(a_-a_+-\frac{1}{2}) \\ H\psi_n &= \hbar\omega(n+\frac{1}{2})\psi_n \\ \psi_0 &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}, \quad \psi_1 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}xe^{-\frac{m\omega}{2\hbar}x^2} \end{aligned}$$

- Relativisitic energy momentum

$$E = \sqrt{p^2c^2 + (mc^2)^2}$$

- Time dependence of an expectation value

$$\frac{d\langle Q\rangle}{dt} = \frac{i}{\hbar}\langle [H, Q]\rangle + \langle \frac{\partial Q}{\partial t}\rangle$$

- Three dimensional infinite cubical well

$$\psi_{n_x,n_y,n_z}(x, y, z) = \left(\frac{2}{a}\right)^{3/2} \sin\left(\frac{n_x\pi}{a}x\right) \sin\left(\frac{n_y\pi}{a}y\right) \sin\left(\frac{n_z\pi}{a}z\right)$$

$$E^0_{n_x,n_y,n_z} = \frac{\pi^2\hbar^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2)$$

- Spin 1/2

$$S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ S_y = \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ S_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \left\langle \frac{1}{2}, \frac{1}{2}(z') \mid R_y(\theta) \mid \frac{1}{2}, \frac{1}{2}(z)\right\rangle & \left\langle \frac{1}{2}, \frac{1}{2}(z') \mid R_y(\theta) \mid \frac{1}{2}, -\frac{1}{2}(z)\right\rangle \\ \left\langle \frac{1}{2}, -\frac{1}{2}(z') \mid R_y(\theta) \mid \frac{1}{2}, \frac{1}{2}(z)\right\rangle & \left\langle \frac{1}{2}, -\frac{1}{2}(z') \mid R_y(\theta) \mid \frac{1}{2}, -\frac{1}{2}(z)\right\rangle \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

- Virial Theorem

$$2\langle T\rangle = \langle \mathbf{r} \cdot \nabla V\rangle$$

- Generators

$$e^{i(\sigma\cdot\hat{n})\phi/2} = \cos(\phi/2) + i(\hat{n}\cdot\sigma)\sin(\phi/2)$$

- Boundary conditions for $V(x) = \alpha\delta(x)$

$$\psi(x) \ \text{ continuous,}$$

$$\Delta\left(\frac{d\psi}{dx}\right) = \frac{2m\alpha}{\hbar^2}\psi(0)$$