Physics 443 Prelim #1 with solutions
March 7, 2008

Each problem is worth 34 points.

1. **Harmonic Oscillator**
   Consider the Hamiltonian for a simple harmonic oscillator
   \[
   H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2
   \]

   (a) Use dimensional analysis to estimate the ground state energy and the characteristic size of the ground state wave function in terms of \(m, \hbar, \) and \(\omega.\) (That is, determine the characteristic length \(l_0\) and energy \(E_0\).)
   
   [Let \(x = l_0z\) and \(E = \epsilon E_0\), where \(l_0\) and \(E_0\) have dimensions of length and energy respectively. Substitution into Schrodinger’s equation gives]
   \[
   \left(-\frac{\hbar^2}{2ml_0^2} \frac{d^2}{dz^2} + \frac{1}{2}m\omega^2l_0^2z^2\right) \psi = \epsilon E_0 \psi
   \]
   Multiply through by \(2ml_0^2/\hbar^2\) to get
   \[
   \left(-\frac{d^2}{dz^2} + m^2\omega^2 \frac{l_0^4}{\hbar^2} z^2\right) \psi = \epsilon \frac{2ml_0^2}{\hbar^2} E_0 \psi
   \]
   Define \(l_0\) so that the coefficient of \(z^2\psi\) on the left hand side is 1 and \(E_0\) so that the coefficient of \(\epsilon \psi\) on the right hand side is 1. Then
   \[
   l_0 = \sqrt{\frac{\hbar}{m\omega}} \quad \text{and} \quad E_0 = \frac{1}{2}h\omega.
   \]

   (b) At \(t = 0\) a particle in the harmonic oscillator potential has as its wave function an even mixture of the first two stationary states with energies \(\frac{1}{2}h\omega\) and \(\frac{3}{2}h\omega\).

   \[\Psi(x,0) = A[\psi_0(x) + \psi_1(x)]\]

   Compute \(A\) and find \(\Psi(x,t)\) and \(|\Psi(x,t)|^2\). Give your answers in terms of \(\psi_0\) and \(\psi_1\).

   [\(A\) is the normalization constant.]

   \[
   \langle \psi | \psi \rangle = 1
   = |A|^2[\langle \psi_0 | \psi_0 \rangle + \langle \psi_0 | \psi_1 \rangle + \langle \psi_1 | \psi_0 \rangle + \langle \psi_1 | \psi_1 \rangle]
   = |A|^2[1 + 0 + 0 + 1]
   \Rightarrow A = \frac{1}{\sqrt{2}}
   \]

   1
\[ \Psi(x, t) = \frac{1}{\sqrt{2}} \left[ \psi_0(x) e^{-iE_0/\hbar t} + \psi_1(x) e^{-iE_1/\hbar t} \right] = \frac{1}{\sqrt{2}} \left[ \psi_0(x) e^{-i\omega t/2} + \psi_1(x) e^{-3i\omega t/2} \right] \]

\[ |\Psi(x, t)|^2 = \frac{1}{2} \left[ 1 + \psi_0^\ast \psi_1 e^{-i\omega t} + \psi_1^\ast \psi_0 e^{i\omega t} \right] \]

(c) Compute \( \langle x \rangle \) using \( x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \). What is the angular frequency of the oscillation?

\[ |\langle x \rangle| = \langle \psi | x | \psi \rangle \]
\[ = \frac{1}{2} \left( \langle \psi_0 | x | \psi_0 \rangle + \langle \psi_1 | x | \psi_1 \rangle + \langle \psi_1 | x | \psi_0 \rangle e^{i\omega t} + \langle \psi_0 | x | \psi_1 \rangle e^{-i\omega t} \right) \]
\[ = \frac{1}{2} \left( \langle \psi_1 | x | \psi_0 \rangle e^{i\omega t} + \langle \psi_0 | x | \psi_1 \rangle e^{-i\omega t} \right) \]
\[ = \langle \psi_1 | x | \psi_0 \rangle \cos(\omega t) \]

The last step follows from the fact that the wave functions \( \psi_1 \) and \( \psi_0 \) are real and \( x \) is an Hermitian operator. Finally

\[ \langle \psi_1 | x | \psi_0 \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \psi_1 | a_+ + a_- | \psi_0 \rangle \]
\[ = \sqrt{\frac{\hbar}{2m\omega}} \langle \psi_1 | a_+ | \psi_0 \rangle \]
\[ = \sqrt{\frac{\hbar}{2m\omega}} \]

The last step follows from the third equation for the one dimensional harmonic oscillator on the formula sheet. So

\[ \langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} \cos(\omega t). \]

The angular frequency of the oscillation is \( \omega \).]

(d) If you measured the energy of this particle, what values might you get and what is the probability of getting each of them?

[A measurement of the energy of the particle would give either \( \frac{1}{2} \hbar \omega \) or \( \frac{3}{2} \hbar \omega \) with equal likelihood.]
(e) What is the expectation value of $H$?

$$\langle H \rangle = \frac{1}{2}[\langle \psi_0 \mid H \mid \psi_0 \rangle + \langle \psi_1 \mid H \mid \psi_1 \rangle] = \hbar \omega$$

2. WKB

Consider the infinite square well with a sloped floor

$$V(x) = \begin{cases} 
\infty & \text{for } (x < 0), \\
kx & \text{for } (0 < x < a), \\
\infty & \text{for } (x > a)
\end{cases}$$

(a) If the well is narrow (small $a$) and $k$ is small, the turning points for the ground state will be at $x = 0$ and $x = a$. If the well is very broad, the right turning point for the ground state will occur along the floor, at $x < a$. Assuming that the turning points are at $x = 0$ and $x < a$, use the WKB approximation to find the energy of the ground state and the turning point.

[The turning point $x_t = E/k$. The quantization condition for a well with one infinite vertical wall is]

$$(n - \frac{1}{4}) \pi \hbar = \int_{x_1}^{x_2} p dx$$

$$= \int_{0}^{x_t} \sqrt{2m(E - kx)} dx$$

$$= \sqrt{2mk} \int_{0}^{x_t} \left( \frac{E}{k} - x \right)^{1/2} dx$$

$$= -\frac{2}{3} \sqrt{2mk} \left( \frac{E}{k} - x \right)^{3/2} |_{x_1}^{x_t}$$

$$= \frac{2}{3} \sqrt{2mk} \left( \frac{E}{k} \right)^{3/2}$$

$$\Rightarrow E = \left( \frac{33}{24} \frac{\pi \hbar}{\sqrt{2m}} \right)^{2/3}$$

(b) Now assume that the turning points are at $x = 0$ and $x = a$ and use the WKB approximation to write an expression that determines the energy levels of the system. (You do not need to solve for $E_n$.) [Now we use the quantization condition for two infinite]
vertical walls.

\[
\pi h = \int_{x_1}^{x_2} p\,dx \\
= \int_{0}^{a} \sqrt{2m(E - kx)}\,dx \\
= \sqrt{2mk} \int_{0}^{a} \left( \frac{E}{k} - x \right)^{\frac{3}{2}}\,dx \\
= \frac{2}{3} \sqrt{2mk} \left( \frac{E}{k} \right)^{3/2} - \left( \frac{E}{k} - a \right)^{3/2} \\
\pi h = \frac{2}{3} \sqrt{2mk} \left( \frac{E}{k} \right)^{3/2} - \left( \frac{E}{k} - a \right)^{3/2} \\
\]

3. Spin 1/2

Suppose that a spin-1/2 particle is in the state

\[\chi = \begin{pmatrix} a \\ b \end{pmatrix},\]

where \(a = \cos \alpha\) and \(b = \sin \alpha\) are real and the state is normalized.

(a) What are the probabilities of getting \(+\hbar/2\) and \(-\hbar/2\), if you measure \(S_z\) and \(S_x\)?

[The probability for getting \(+\hbar/2\) if you measure along the \(z\) direction is

\[|\chi_+^\dagger \chi|^2 = |(1 \ 0) \left( \begin{array}{c} a \\ b \end{array} \right)|^2 = |a|^2\]

and the probability for \(-\hbar/2\) is \(|b|^2\). The probability for getting \(+\hbar/2\) if you measure along the \(x\) direction is

\[|\chi_+^{(x)} \chi|^2 = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \left( \begin{array}{c} a \\ b \end{array} \right) |^2 = \frac{1}{2} |a + b|^2.\]

\(\chi_+^{(x)}\) are the eigenvectors of \(S_x\) with eigenvalues \(\pm \frac{1}{2}\hbar\). The probability for \(-\hbar/2\) is \(\frac{1}{2} |a - b|^2\).]

(b) In a coordinate system rotated by an angle \(\theta\) about the \(y\)-axis so that \(z \rightarrow z'\) and \(x \rightarrow x'\), what are the the probabilities of getting \(+\hbar/2\) and \(-\hbar/2\), if you measure \(S_{z'}\) and \(S_{x'}\)?
In the rotated coordinate system

\[
\chi' = \begin{pmatrix}
\cos(\theta/2) & -\sin(\theta/2) \\
\sin(\theta/2) & \cos(\theta/2)
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
= \begin{pmatrix}
a' \\
b'
\end{pmatrix}
= \begin{pmatrix}
 a \cos(\theta/2) - b \sin(\theta/2) \\
b \cos(\theta/2) + a \sin(\theta/2)
\end{pmatrix}
\]

Now the probability of getting $+\hbar/2$ if you measure along the $+z'$ direction is $|a'|^2$ and the probability of getting $-\hbar/2$ is $|b'|^2$. The probability of getting $\pm \hbar/2$ if you measure along the $+x'$ axis is $\frac{1}{2}|a' \pm b'|^2$. 
Formulae and Tables

\[ H\Psi = i\hbar \frac{\partial \Psi}{\partial t} \]
\[ -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} = i\hbar \frac{\partial}{\partial t} \Psi(x, t) \]

• WKB

\[ \int_{x_1}^{x_2} p(x) dx = \left( n - \frac{1}{2} \right) \pi \hbar \] (no infinite vertical walls)
\[ \int_{x_1}^{x_2} p(x) dx = \left( n - \frac{1}{4} \right) \pi \hbar \] (1 infinite vertical wall)
\[ \int_{x_1}^{x_2} p(x) dx = n\pi \hbar \] (2 infinite vertical walls)

\[ \psi(x) = \frac{A}{\sqrt{p}} \exp\left( \frac{i}{\hbar} \int_{x_1}^{x} p(x') dx' \right) + \frac{B}{\sqrt{p}} \exp\left( -\frac{i}{\hbar} \int_{x_1}^{x} p(x') dx' \right) \]

\[ \psi(x) = \frac{C}{\sqrt{|p|}} \exp\left( \frac{1}{\hbar} \int_{x_1}^{x} |p(x')| dx' \right) + \frac{D}{\sqrt{|p|}} \exp\left( -\frac{1}{\hbar} \int_{x_1}^{x} |p(x')| dx' \right) \]

• One dimensional harmonic oscillator

\[ H = \frac{\hat{p}^2}{2m} + \frac{1}{2} \frac{m\omega^2}{\hbar} \hat{x}^2 \]
\[ a_{\pm} = \frac{1}{\sqrt{2\hbar m\omega}} (\mp i\hat{p} + m\omega \hat{x}) \]
\[ a_+\psi_n = \sqrt{(n + 1)} \psi_{n+1} \]
\[ a_-\psi_n = \sqrt{n} \psi_{n-1} \]
\[ [a_-, a_+] = 1 \]
\[ H = \hbar \omega (a_- a_+ - \frac{1}{2}) \]
\[ H\psi_n = \hbar \omega (n + \frac{1}{2}) \psi_n \]
\[ \psi_0 = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} e^{-\frac{m\omega}{2\pi} x^2}, \quad \psi_1 = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}} x e^{-\frac{m\omega}{2\pi} x^2} \]
• Relativistic energy momentum

\[ E = \sqrt{p^2 c^2 + (mc^2)^2} \]

• Time dependence of an expectation value

\[ \frac{d\langle Q \rangle}{dt} = i\frac{\hbar}{\hbar}\langle [H, Q] \rangle + \langle \frac{\partial Q}{\partial t} \rangle \]

• Three dimensional infinite cubical well

\[ \psi_{n_x,n_y,n_z}(x,y,z) = \left(\frac{2}{a}\right)^{3/2} \sin\left(\frac{n_x\pi}{a}x\right) \sin\left(\frac{n_y\pi}{a}y\right) \sin\left(\frac{n_z\pi}{a}z\right) \]

\[ E_{n_x,n_y,n_z}^0 = \frac{\pi^2\hbar^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2) \]

• Spin 1/2

\[ S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[
\begin{pmatrix}
\left\langle \frac{1}{2}, \frac{1}{2} \middle| R_y(\theta) \middle| \frac{1}{2}, \frac{1}{2} \right\rangle & \left\langle \frac{1}{2}, \frac{1}{2} \middle| R_y(\theta) \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle \\
\left\langle \frac{1}{2}, -\frac{1}{2} \middle| R_y(\theta) \middle| \frac{1}{2}, \frac{1}{2} \right\rangle & \left\langle \frac{1}{2}, -\frac{1}{2} \middle| R_y(\theta) \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle \\
\end{pmatrix}
= \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta \\
\end{pmatrix}
\]

• Virial Theorem

For stationary states

\[ 2\langle T \rangle = \langle r \cdot \nabla V \rangle \]

• Generators

\[ e^{i(\sigma \cdot \hat{n})\phi/2} = \cos(\phi/2) + i(\hat{n} \cdot \sigma) \sin(\phi/2) \]

• Boundary conditions for \( V(x) = \alpha \delta(x) \)

\[ \psi(x) \text{ continuous,} \]

\[ \Delta \left( \frac{d\psi}{dx} \right) = \frac{2m\alpha}{\hbar^2} \psi(0) \]