P443 HW #1
http://www.lns.cornell.edu/~dlr/teaching/p443
Due January 30, 2008

1. Griffiths, 1.9. A particle of mass m is in the state

$$\Psi(x,t) = Ae^{-a[(mx^2/\hbar)+it]},$$

where A and a are positive real constants

   (a) Find A.

   (b) For what potential energy function $V(x)$ does Ψ satisty the Shrodinger equation?

   (c) Calculate the expectation values of x, x^2, p, and p^2.

   (d) Find σ_x and σ_p. Is their product consistent with the uncertainty principle?

2. Griffiths, 1.16. Show that

$$\frac{d}{dt}\int_{-\infty}^{\infty} \Psi_1^* \Psi_2 dx = 0$$

for any two (normalizable) solutions to the Schrodinger equation, Ψ_1 and Ψ_2.

3. Griffiths, 2.5. A particle in the infinite square well has as its initial wave function an even mixture of the first two stationary states:

$$\Psi(x,0) = A[\psi_1(x) + \psi_2(x)].$$

   (a) Normalize $\Psi(x,0)$. (That is, find A. This is very easy, if you exploit the orthonormality of ψ_1 and ψ_2. Recall that, having normalized Ψ at $t = 0$, you can rest assured that it *stays* normalized-if you doubt this, check it explicity after doing part (b).

   (b) Find $\Psi(x,t)$ and $|\Psi(x,t)|^2$. Express the latter as a sinusoidal function of time, as in Example 2.1. To simplify the result, let $\omega = \pi^2\hbar/2ma^2$.

(c) Compute $\langle x \rangle$. Notice that it oscillates in time. What is the angular frequency of the oscillation? What is the amplitude of the oscillation? (It had better be $\leq a/2$.)

(d) Compute $\langle p \rangle$

(e) If you measured the energy of this particle, what values might you get, and what is the probability of getting each of them? Find the expectation value of H. How does it compare with E_1 and E_2?

4. Griffiths, 2.21 A free particle has the initial wave function

$$\Psi(x,0) = Ae^{-a|x|},$$

where A and a are positive real constants.

(a) Normalize $\Psi(x,0)$.

(b) Find $\phi(k)$.

(c) Construct $\Psi(x,t)$ in the form of an integral.

(d) Discuss the limiting cases (a very large, and a very small).

5. Griffiths, 2.22 **The gaussian wave packet.** A free particle has the initial wave function

$$\psi(x,0) = Ae^{-ax^2},$$

where A and a are constants (a is real and positive).

(a) Normalize $\Psi(x,0)$.

(b) Find $\Psi(x,t)$. *Hint*: Integrals of the form

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx)}dx$$

can be handled by "completing the square": Let $y \equiv \sqrt{a}[x + (b/2a)]$, and note that $(ax^2 + bx) = y^2 - (b^2/4a)$. *Answer* :

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-ax^2/[1+(2i\hbar at/m)]}}{\sqrt{1+(2i\hbar at/m)}}.$$

(c) Find $|\Psi(x,t)|^2$. Express your answer in terms of the quantity

$$w \equiv \sqrt{\frac{a}{1+(2\hbar a t/m)^2}}.$$

Sketch $|\Psi|^2$ (as a function of x) at $t=0$ and again for some very large t. Qualitatively, what happens to $\Psi|^2$, as times goes on?

(d) Find $\langle x\rangle, \langle p\rangle, \langle x^2\rangle, \langle p^2\rangle, \sigma_x$, and σ_p. *Partial answer* : $\langle p^2\rangle = a\hbar^2$, but it may take some algebra to reduce it to this simple form.

(e) Does the uncertainty principle hold? At what time t does the system come closest to the uncertainty limit?

6. Current Vector

   Find the current density carried by a plane wave Ae^{ikx} in one dimension, showing that it is in fact what one would expect from the formula $\rho\mathbf{v}$, and verify that it satisfies the equation of continuity.

7. Commutators

   Prove the following:

$$[\hat{A},\hat{B}] = -[\hat{B},\hat{A}]$$
$$[\hat{A}+\hat{B},\hat{C}] = [\hat{A},\hat{C}] + [\hat{B},\hat{C}]$$
$$[a,\hat{A}] = 0$$
$$[a\hat{A},\hat{B}] = a[\hat{A},\hat{B}]$$
$$[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$$

   where a is a constant number.

8. Consider the 1-dimensional hamiltonian $H = \frac{p^2}{2m} + V(x)$, $[x,p] = i\hbar$, and $H|\,\psi_n\rangle = E_n|\,\psi_n\rangle$.

   (a) Show that $\langle\psi_n\mid p\mid\psi_{n'}\rangle = \alpha\,\langle\psi_n\mid x\mid\psi_{n'}\rangle$. Calculate $\alpha = \alpha(E_n - E_{n'})$.

   (b) Derive the "sum rule"

$$\sum_{n'}(E_n - E_{n'})^2|\,\langle\psi_n\mid x\mid\psi_{n'}\rangle\,|^2 = \frac{\hbar^2}{m^2}\left\langle\psi_n\mid p^2\mid\psi_n\right\rangle.$$