1. Griffiths, 1.9. A particle of mass $m$ is in the state

$$\Psi(x, t) = Ae^{-a[(mx^2/\hbar)+it]}$$

where $A$ and $a$ are positive real constants

(a) Find $A$.

(b) For what potential energy function $V(x)$ does $\Psi$ satisfy the Schrödinger equation?

(c) Calculate the expectation values of $x, x^2, p,$ and $p^2$.

(d) Find $\sigma_x$ and $\sigma_p$. Is their product consistent with the uncertainty principle?

2. Griffiths, 1.16. Show that

$$\frac{d}{dt} \int_{-\infty}^{\infty} \Psi_1^* \Psi_2 dx = 0$$

for any two (normalizable) solutions to the Schrödinger equation, $\Psi_1$ and $\Psi_2$.

3. Griffiths, 2.5. A particle in the infinite square well has as its initial wave function an even mixture of the first two stationary states:

$$\Psi(x, 0) = A[\psi_1(x) + \psi_2(x)]$$

(a) Normalize $\Psi(x, 0)$. (That is, find $A$. This is very easy, if you exploit the orthonormality of $\psi_1$ and $\psi_2$. Recall that, having normalized $\Psi$ at $t = 0$, you can rest assured that it stays normalized-if you doubt this, check it explicitly after doing part (b).

(b) Find $\Psi(x, t)$ and $|\Psi(x, t)|^2$. Express the latter as a sinusoidal function of time, as in Example 2.1. To simplify the result, let $\omega = \pi^2 \hbar / 2ma^2$. 

1
(c) Compute $\langle x \rangle$. Notice that it oscillates in time. What is the angular frequency of the oscillation? What is the amplitude of the oscillation? (It had better be $\leq a/2$.)

(d) Compute $\langle p \rangle$

(e) If you measured the energy of this particle, what values might you get, and what is the probability of getting each of them? Find the expectation value of $H$. How does it compare with $E_1$ and $E_2$?

4. Griffiths, 2.21 A free particle has the initial wave function

$$\Psi(x, 0) = Ae^{-a|x|},$$

where $A$ and $a$ are positive real constants.

(a) Normalize $\Psi(x, 0)$.

(b) Find $\phi(k)$.

(c) Construct $\Psi(x, t)$ in the form of an integral.

(d) Discuss the limiting cases ($a$ very large, and $a$ very small).

5. Griffiths, 2.22 The Gaussian wave packet. A free particle has the initial wave function

$$\psi(x, 0) = Ae^{-ax^2},$$

where $A$ and $a$ are constants ($a$ is real and positive).

(a) Normalize $\Psi(x, 0)$.

(b) Find $\Psi(x, t)$. Hint: Integrals of the form

$$\int_{-\infty}^{\infty} e^{-ax^2+bx} dx$$

can be handled by "completing the square": Let $y \equiv \sqrt{a}[x + (b/2a)]$, and note that $(ax^2 + bx) = y^2 - (b^2/4a)$. Answer:

$$\Psi(x, t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-ax^2/[1+(2\hat{\imath} \hat{m})]}}{\sqrt{1+(2\hat{\imath} \hat{m})}}.$$
(c) Find $|\Psi(x,t)|^2$. Express your answer in terms of the quantity

$$w \equiv \sqrt{\frac{a}{1 + (2\hbar a/m)^2}}.$$

Sketch $|\Psi|^2$ (as a function of $x$) at $t = 0$ and again for some very large $t$. Qualitatively, what happens to $|\Psi|^2$, as times goes on?

(d) Find $\langle x \rangle, \langle p \rangle, \langle x^2 \rangle, \sigma_x,$ and $\sigma_p$. **Partial answer**: $\langle p^2 \rangle = \hbar^2$, but it may take some algebra to reduce it to this simple form.

(e) Does the uncertainty principle hold? At what time $t$ does the system come closest to the uncertainty limit?

6. Current Vector

Find the current density carried by a plane wave $Ae^{ikx}$ in one dimension, showing that it is in fact what one would expect from the formula $\rho v$, and verify that it satisfies the equation of continuity.

7. Commutators

Prove the following:

$$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$$
$$[\hat{A} + \hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]$$
$$[a, \hat{A}] = 0$$
$$[a\hat{A}, \hat{B}] = a[\hat{A}, \hat{B}]$$
$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$$

where $a$ is a constant number.

8. Consider the 1-dimensional hamiltonian $H = \frac{p^2}{2m} + V(x)$, $[x,p] = i\hbar$, and $H|\psi_n\rangle = E_n|\psi_n\rangle$.

(a) Show that $\langle \psi_n | p | \psi_{n'} \rangle = \alpha \langle \psi_n | x | \psi_{n'} \rangle$. Calculate $\alpha = \alpha(E_n - E_{n'})$.

(b) Derive the "sum rule"

$$\sum_{n'} (E_n - E_{n'})^2 |\langle \psi_n | x | \psi_{n'} \rangle|^2 = \hbar^2 \langle \psi_n | p^2 | \psi_n \rangle.$$