Physics 443, Solutions to PS 13

1 Griffiths 11.12

In this problem we study the Yukawa potential. From Griffiths Eq. 11.81, we see that

$$f(\theta) = \frac{-2m\beta}{\hbar^2(\mu^2+q^2)},$$

where $q = 2k\sin(\theta/2)$. We have that

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \left(\frac{2m\beta}{\hbar^2}\right)^2 \frac{1}{(\mu^2+q^2)^2} \\
\sigma &= \int \frac{d\sigma}{d\Omega} d\Omega \\
&= 2\pi \left(\frac{2m\beta}{\hbar^2}\right)^2 \frac{1}{\mu^4} \int_0^\pi \frac{2\sin(\theta/2)\cos(\theta/2)d\theta}{[1+((2k/\mu)\sin(\theta/2)^2])^2} \\
&= \pi \left(\frac{2m\beta}{\hbar^2}\right)^2 \frac{1}{(\mu k)^2} \left[1 - \frac{1}{1+(2k/\mu)^2}\right] \\
&= \pi \left(\frac{4m\beta}{\mu\hbar}\right)^2 \frac{1}{(\mu\hbar)^2 + 8mE}
\end{aligned}$$

Where $E = \hbar^2 k^2/2m$ is the energy.

2 Square Well

For a spherical square well potential we have

$$\begin{aligned}
f(\theta) &= \frac{-2m}{\hbar^2 q} \int rV(r)\sin(qr)\ dr \\
&= \frac{-2mV_o}{\hbar^2 q} \int_0^a r\sin(qr)\ dr \\
&= \frac{2mV_o}{\hbar^2 q^3} (\sin(qa) - qa\cos(qa)) \\
\frac{d\sigma}{d\Omega} &= \left(\frac{2mV_0 a^3}{\hbar^2}\right)^2 \left[\frac{\sin(qa) - qa\cos(qa)}{(qa)^3}\right]^2
\end{aligned}$$

For $qa \ll 1$,

$$\frac{d\sigma}{d\Omega} \approx \left(\frac{2mV_0 a^3}{\hbar^2}\right)^2 \left[\frac{qa - \frac{1}{3!}(qa)^3 + \frac{1}{5!}(qa)^5 + ... - (qa)(1 - \frac{1}{2}(qa)^2 + \frac{1}{4!}(qa)^4 + ...)}{(qa)^3}\right]^2$$

$$
\begin{aligned}
&\approx \left(\frac{2mV_0a^3}{\hbar^2}\right)^2 \left[\frac{\frac{1}{3}(qa)^3 - \frac{4}{5!}(qa)^5)}{(qa)^3}\right]^2 \\
&\approx \left(\frac{2mV_0a^3}{3\hbar^2}\right)^2 \left[1 - \frac{1}{10}(qa)^2)\right]^2 \\
&\approx \left(\frac{2mV_0a^3}{3\hbar^2}\right)^2 \left[1 - 2\frac{1}{10}(qa)^2)\right]^2
\end{aligned}
$$

where in the very last step we square and expand $(1+\epsilon)^2 \approx (1+2\epsilon)$ Then

$$
\begin{aligned}
\sigma = \int \frac{d\sigma}{d\Omega} &\approx \left(\frac{2mV_0a^3}{3\hbar^2}\right)^2 \int \left(1 - \frac{1}{5}(qa)^2\right) \sin\theta d\theta d\phi \\
&\approx \left(\frac{2mV_0a^3}{3\hbar^2}\right)^2 \int \left(1 - \frac{1}{5}(2k\sin\frac{\theta}{2}a)^2\right) \sin\theta d\theta d\phi \\
&\approx \left(\frac{2mV_0a^3}{3\hbar^2}\right)^2 \int \left(1 - \frac{2}{5}(ka)^2(1-\cos\theta)\right) \sin\theta d\theta d\phi \\
&\approx \left(\frac{2mV_0a^3}{3\hbar^2}\right)^2 4\pi \left(1 - \frac{2}{5}(ka)^2\right)
\end{aligned}
$$

To get the total cross section in the high energy limit we integrate the exact result

$$
\int \frac{d\sigma}{d\Omega} d\Omega = \left(\frac{2mV_0a^3}{\hbar^2}\right)^2 \left[\frac{\sin(qa) - qa\cos(qa)}{(qa)^3}\right]^2 \sin\theta d\theta d\phi
$$

Let $x = 2ka\sin\frac{\theta}{2}$. Then $dx = ka\cos\frac{\theta}{2}d\theta$

$$
\begin{aligned}
\sigma &= \left(\frac{2mV_0a^3}{\hbar^2}\right)^2 \int \left[\frac{\sin(x) - x\cos(x)}{x^3}\right]^2 \frac{xdx}{(ka)^2} d\phi \\
&= \left(\frac{2mV_0a^3}{\hbar^2}\right)^2 \int \frac{(\sin(x) - x\cos(x))^2}{x^5} \frac{dx}{(ka)^2)} d\phi \\
&= \frac{2\pi}{(ka)^2} \left(\frac{2mV_0a^3}{\hbar^2}\right)^2 \int_0^{ka} \frac{(\sin(x) - x\cos(x))^2}{x^5} dx \\
&= \frac{2\pi}{(ka)^2} \left(\frac{2mV_0a^3}{\hbar^2}\right)^2 \int_0^{ka} \frac{(\sin(x) - x\cos(x))^2}{x^5} dx
\end{aligned}
$$

Then using the hint

$$\begin{aligned}\sigma &= -\frac{2\pi}{(ka)^2}\left(\frac{2mV_0a^3}{\hbar^2}\right)^2\frac{1}{4}\left[\frac{(\sin x - x\cos x)^2}{x^4}+\frac{\sin^2 x}{x^2}\right]_0^{ka}\\ &= \frac{2\pi}{(ka)^2}\left(\frac{2mV_0a^3}{\hbar^2}\right)^2\frac{1}{4}\end{aligned}$$

Note that in when $x = 0$, that the first term in the square brackets goes to 0 and the second term is 1. And in the high energy limit $ka \gg 1$, both terms are zero.

Dirac equation

The Dirac equation is

$$i\hbar\frac{\partial\psi}{\partial t} = [c\alpha\cdot\mathbf{p} + \beta mc^2]\psi$$

where

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

and

$$\psi(\mathbf{r}, t) = \begin{pmatrix} \psi_1(\mathbf{r}, t) \\ \psi_2(\mathbf{r}, t) \end{pmatrix}$$

and ψ_1 and ψ_2 are each two component spinors. If we assume a time dependence $\psi(\mathbf{r}, t) = \psi(\mathbf{r})e^{-iEt/\hbar}$, then the Dirac equation becomes

$$E\psi = [c\alpha\cdot\mathbf{p} + \beta mc^2]\psi \tag{1}$$

In order to include a magnetic field we replace $\mathbf{p} \to \mathbf{p} - q\mathbf{A}$.

(a) Show that in the nonrelativistic limit

$$E_S\psi_1 = \frac{[\sigma\cdot(\mathbf{p} - q\mathbf{A})][\sigma\cdot(\mathbf{p} - q\mathbf{A})]}{2m}\psi_1 \tag{2}$$

where $E_S = E - mc^2$.

[We can rewrite Equation 1

$$
\begin{aligned}
E\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} &= [c\alpha\cdot\mathbf{p}+\beta mc^2]\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\\
\Rightarrow \begin{matrix}E\psi_1\\ E\psi_2\end{matrix} &= \begin{matrix}c\sigma\cdot\mathbf{p}\psi_2+mc^2\psi_1\\ c\sigma\cdot\mathbf{p}\psi_1-mc^2\psi_2\end{matrix}
\end{aligned}
$$

Solve the last equation for

$$\psi_2=\frac{c\sigma\cdot\mathbf{p}}{E+mc^2}\psi_1$$

and substitute into the next to last equation to get

$$
\begin{aligned}
\psi_1 &= \frac{(c\sigma\cdot\mathbf{p})}{(E-mc^2)}\frac{(c\sigma\cdot\mathbf{p})}{(E+mc^2)}\psi_1\\
&= \frac{(c\sigma\cdot\mathbf{p})}{E_S}\frac{(c\sigma\cdot\mathbf{p})}{(E_S+2mc^2)}\psi_1\\
\rightarrow E_S\psi_1 &= (c\sigma\cdot\mathbf{p})\frac{(c\sigma\cdot\mathbf{p})}{2mc^2(1+\frac{E_S}{2mc^2})}\psi_1\\
E_S\psi_1 &\approx \frac{(\sigma\cdot\mathbf{p})(\sigma\cdot\mathbf{p})}{2m}\psi_1
\end{aligned}
$$

Then with the substitution $\mathbf{p}\rightarrow\mathbf{p}-q\mathbf{A}$, we get

$$E_S\psi_1\approx\frac{\sigma\cdot(\mathbf{p}-q\mathbf{A})\sigma\cdot(\mathbf{p}-q\mathbf{A})}{2m}\psi_1]$$

(b) Prove the identity

$$\sigma\cdot\mathbf{A}\sigma\cdot\mathbf{B}=\mathbf{A}\cdot\mathbf{B}+i\sigma\cdot\mathbf{A}\times\mathbf{B}$$

for any vectors $\mathbf{A}$ and $\mathbf{B}$.

[

$$
\begin{aligned}
\sigma\cdot\mathbf{A}\sigma\cdot\mathbf{B} &= \sum_i\sum_j \sigma_i A_i\sigma_j A_j\\
&= \sum_i\sum_j \sigma_i\sigma_j A_i B_j\\
&= \sum_i\sum_j i\epsilon_{ijk}\sigma_k A_i B_j+\delta_{ij}A_iB_j\\
&= i\sigma\cdot(\mathbf{A}\times\mathbf{B})+\mathbf{A}\cdot\mathbf{B}]
\end{aligned}
$$

(c) Show that

$$(\mathbf{p} - q\mathbf{A}) \times (\mathbf{p} - q\mathbf{A}) = iq\hbar\mathbf{B}$$

[Remember $\mathbf{p} = -i\hbar\nabla$ is an operator that does not commute with $\mathbf{A}$

$$\begin{aligned}
(\mathbf{p} - q\mathbf{A}) \times (\mathbf{p} - q\mathbf{A}) &= \mathbf{p} \times \mathbf{p} - q(\mathbf{p} \times \mathbf{A}) - q(\mathbf{A} \times \mathbf{p}) + q^2\mathbf{A} \times \mathbf{A} \\
&= -q(\mathbf{p} \times \mathbf{A})\{\psi\} - q(\mathbf{A} \times \mathbf{p})\{\psi\} \\
&= i\hbar q \left(\nabla \times \mathbf{A}\{\psi\} - \mathbf{A} \times \nabla\{\psi\} + \mathbf{A} \times \nabla\{\psi\}\right) \\
&= i\hbar q \nabla \times \mathbf{A} \\
&= i\hbar q\mathbf{B}]
\end{aligned}$$

(d) And that Equation 2 becomes

$$\left[\frac{(\mathbf{p} - q\mathbf{A})^2}{2m} - \mu \cdot \mathbf{B}\right]\psi_1 = E_S\psi_1$$

where $\mu = \frac{q}{m}\mathbf{s}$ and $\mathbf{s} = \frac{\hbar}{2}\sigma$.

[Begin with Equation 2 and use the identity from part (b)

$$\begin{aligned}
E_S\psi_1 &= \frac{[\sigma\cdot(\mathbf{p} - q\mathbf{A})][\sigma\cdot(\mathbf{p} - q\mathbf{A})]}{2m}\psi_1 \\
&= \frac{(\mathbf{p} - q\mathbf{A})^2 + \sigma\cdot[(\mathbf{p} - q\mathbf{A}) \times (\mathbf{p} - q\mathbf{A})]}{2m}\psi_1 \\
&= \frac{(\mathbf{p} - q\mathbf{A})^2 - \hbar q\sigma \cdot \mathbf{B}}{2m}\psi_1 \\
&= \left[\frac{(\mathbf{p} - q\mathbf{A})^2}{2m} - \mu\sigma \cdot \mathbf{B}\right]\psi_1]
\end{aligned}$$