Physics 443, Solutions to PS 1[1]

1. Griffiths 1.9

   For $\Phi(x,t) = A\exp[-a(\frac{mx^2}{\hbar} + it)]$, we need that $\int_{-\infty}^{+\infty} |\Phi(x,t)|^2 dx = 1$. Using the known result of a Gaussian intergral $\int_{-\infty}^{+\infty} \exp[-ax^2]dx = \sqrt{\pi/a}$, we find that:

$$A = \sqrt{\sqrt{\frac{2am}{\pi\hbar}}}. \tag{1}$$

   The Schrödinger Equation is given by $\mathcal{H}\Phi = i\hbar\frac{\partial\Phi}{\partial t}$, with $\mathcal{H} = (-\hbar^2/2m)\frac{\partial^2}{\partial x^2} + V(x)$. Plugging our Wavefunction into this Equation, we find:

$$\begin{aligned} \frac{-\hbar^2}{2m}(\frac{-2am}{\hbar} + (\frac{2amx}{\hbar})^2) + V = a\hbar, \\ a\hbar - 2a^2mx^2 + V = a\hbar, \\ V(x) = 2ma^2x^2. \end{aligned} \tag{2}$$

   Being odd functions, $\langle x\rangle$ and $\langle p\rangle$ are zero. And using $\int_{-\infty}^{+\infty} x^2\exp[-ax^2]dx = 0.5\sqrt{\pi/a^3}$, we have:

$$\begin{aligned} \langle x^2\rangle = \frac{\hbar}{4am}, \langle p^2\rangle = \hbar am, \\ \sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{1}{2}\sqrt{\frac{\hbar}{am}}, \\ \sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{am\hbar}, \\ \sigma_x\sigma_p = \frac{\hbar}{2}. \end{aligned} \tag{3}$$

2. Griffiths 1.16

$$\frac{d}{dt}\int_{-\infty}^{\infty} \psi_1^*\psi_2 dx = \int_{-\infty}^{\infty} \left(\frac{\partial\psi_1^*}{\partial t}\psi_2 + \psi_1^*\frac{\partial\psi_2}{\partial t}\right) dx \tag{4}$$

[1] Courtesy Shaffique Adam

We use Schrodinger's equation and its complex conjugate to replace the time derivatives with space derivatives and the potential $V(x)$. $V(x)$ is assumed real and independent of time.

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty}\psi_1^*\psi_2 dx &= \int_{-\infty}^{\infty}\left[\frac{i}{\hbar}\left(\frac{-\hbar^2}{2m}\frac{\partial^2\psi_1^*}{\partial x^2}+V\psi_1^*\right)\psi_2-\frac{i}{\hbar}\psi_1^*\left(\frac{-\hbar^2}{2m}\frac{\partial^2\psi_2}{\partial x^2}+V\psi_2\right)\right]dx \\ &= \frac{i}{\hbar}\left(\frac{-\hbar^2}{2m}\right)\int_{-\infty}^{\infty}\left(\frac{\partial^2\psi_1^*}{\partial x^2}\psi_2-\psi_1^*\frac{\partial^2\psi_2}{\partial x^2}\right)dx\end{aligned}$$

Integrate by parts and use the fact that normalizable wave functions are zero at infinity to drop the total derivative. Then

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty}\psi_1^*\psi_2 dx &= \left(\frac{-i\hbar}{2m}\right)\int_{-\infty}^{\infty}\left(\frac{\partial\psi_1^*}{\partial x}\frac{\partial\psi_2}{\partial x}-\frac{\partial\psi_1^*}{\partial x}\frac{\partial\psi_2}{\partial x}\right)dx \\ &= 0\end{aligned}$$

3. Griffiths 2.5

   (a) We have that $\Psi(x,0) = A[\psi_1(x)+\psi_2(x)]$. We know that the states $\psi_n(x) = \sqrt{\frac{2}{L}}\sin k_n x$, with $k_n = n\pi/L$ are normalized, orthogonal and real. Then

$$\begin{aligned}\int_0^L \Psi^*(x,0)\Psi(x,0)dx &= |A|^2\int_0^L\left(|\psi_1|^2+|\psi_2|^2+2\psi_1\psi_2\right)dx \\ &= |A|^2(2) = 1 \\ &\rightarrow\ A = \frac{1}{\sqrt{2}}\end{aligned}$$

   (b)

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_1(x)e^{-i\omega_1 t}+\psi_2(x)e^{-i\omega_2 t}\right) \tag{5}$$

   where $\omega_n = \frac{\hbar k_n^2}{2m}$. Then

$$\begin{aligned}|\Psi(x,t)|^2 &= \frac{1}{2}\left[|\psi_1(x)|^2+|\psi_2(x)|^2+2\psi_1(x)\psi_2(x)\cos(\Delta\omega t)\right] \\ &= \frac{1}{L}\left[\sin^2(k_1x)+\sin^2(k_2x)+2\sin(k_1x)\sin(k_2x)\cos\Delta\omega t\right]\end{aligned}$$

   where $\Delta\omega = \omega_1-\omega_2$

(c)

$$\begin{aligned}
\langle x\rangle &= \int_0^L x|\Psi(x,t)|^2 dx \\
&= \frac{1}{L}\int_0^L \left[x\sin^2(k_1x) + x\sin^2(k_2x) + 2x\sin(k_1x)\sin(k_2x)\cos\Delta\omega t\right] dx \\
&= \left[\frac{L}{2} + \frac{L}{2} + \frac{2}{L}\left(-\frac{1}{(\pi/L)^2} + \frac{1}{(3\pi/L)^2}\right)\cos\Delta\omega t\right] \\
&= L\left[1 - \frac{8}{9\pi^2}\cos\Delta\omega t\right]
\end{aligned}$$

We got the first two terms in the integration by inspection since $\sin^2 kx$ is symmetric about the midpoint of the well. We find the integral of the third term in a table. It is appended at the end of this solution set. The amplitude of the oscillation is

$$\frac{8L}{9\pi^2}$$

(d)

$$\langle p\rangle = m\frac{d}{dt}\langle x\rangle = \frac{8L}{9\pi^2}\Delta\omega\sin\Delta\omega t$$

(e) An energy measurement would yield either E_1 or E_2. The probability of measuring E_1 is equivalent to the probablity that Ψ is in the state ψ_1, namely

$$P_1 = |\int \psi_1(x)\Psi(x,t)dx|^2 = \frac{1}{2}$$

$$\langle H\rangle = \frac{1}{2}(E_1 + E_2)$$

4. Griffiths 2.21

(a) To normalize the wave function set

$$\int_{-\infty}^{\infty} |\Psi(x,0)|^2 \; dx = 1$$

where

$$\Psi(x,0) = Ae^{-a|x|}$$

Then

$$\int_{-\infty}^{\infty} |\Psi(x,0)|^2 \; dx = 1,$$

$$\begin{aligned} &= |A|^2 \left(\int_0^\infty e^{-2ax} + \int_{-\infty}^0 e^{2ax} \right) dx \\ &= |A|^2 \left(2\frac{1}{2a} \right) = 1 \\ &\rightarrow A = \sqrt{a} \end{aligned}$$

(b) The momentum distribution

$$\begin{aligned} \phi(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx} dx \\ \phi(k) &= \sqrt{\frac{a}{2\pi}} \left(\int_{-\infty}^{\infty} e^{-a|x|} e^{-ikx} \right) dx \\ &= \sqrt{\frac{a}{2\pi}} \left(\int_0^\infty e^{-ax-ikx} + \int_{-\infty}^0 e^{ax-ikx} \right) dx \\ &= \sqrt{\frac{a}{2\pi}} \left(\frac{e^{-ax-ikx}}{-a-ik} |_0^\infty + \frac{e^{ax-ikx}}{a-ik} |_{-\infty}^0 \right) \\ &= \sqrt{\frac{a}{2\pi}} \left(\frac{1}{a+ik} + \frac{1}{a-ik} \right) \\ &= \sqrt{\frac{a}{2\pi}} \frac{2a}{a^2+k^2} \end{aligned}$$

(c)

$$\Psi(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(k) e^{i(kx-\omega_k t)} dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sqrt{a^3}}{a^2+k^2} e^{i(kx-\omega_k t)} dk$$

and $\omega_k = \frac{\hbar k^2}{2m}$.

(d) In the limit where $a \Rightarrow \infty$,

$$\Psi(x,t) \rightarrow \frac{1}{\pi\sqrt{a}} \int_{-\infty}^{\infty} e^{i(kx-\omega_k t)} dk$$

which looks like $\frac{2}{\sqrt{a}}\delta(x)$ at $t = 0$ and will spread out in time due to the k dependence of ω. In the limit $a \rightarrow 0$,

$$\Psi(x,t) = \frac{1}{\sqrt{a^3}\pi} \int_{-\infty}^{\infty} \frac{1}{k^2} e^{i(kx-\omega_k t)} dk$$

Small k, (long wavelenghts) will dominate the distribution at $t = 0$.

5. Griffiths 2.22

   We have

$$\Psi(x,0) = \sqrt{\sqrt{\frac{2a}{\pi}}}\exp(-ax^2).$$

The way to solve this problem is take the Fourier transform $\phi(k)$, since we know how to time evolve $\phi(k)$ for a free particle. In particular, we have the following relations

$$\begin{aligned}
\Psi(x,t) &= \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{+\infty} dk\ \phi(k)\exp(-ikx - \frac{i\hbar k^2 t}{2m}) \\
\phi(k) &= \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{+\infty} dx'\ \Psi(x',0)\exp(-ikx').
\end{aligned}$$

Putting these two equations together, one can solve for $\Psi(x,t)$ as

$$\begin{aligned}
\Psi(x,t) &= \frac{1}{2\pi}\int\int\left(\frac{2a}{\pi}\right)^{\frac{1}{4}}\exp(-ax'^{\,2})\exp(-ikx')\exp(ikx)\exp(\frac{-i\hbar k^2 t}{2m})\ dx'dk, \\
&= \frac{1}{2\pi}\left(\frac{2a}{\pi}\right)^{\frac{1}{4}}\int\int dydk\ \exp(-ay^2 - iky + ikx - i\hbar k^2 t/2m).
\end{aligned}$$

This looks like a mess, but it is really just two integrals of the form given in the hint. Performing the integral over dy first, and then dk, we have

$$\begin{aligned}
\Psi(x,t) &= \frac{1}{2\pi}\left(\frac{2a}{\pi}\right)^{\frac{1}{4}}\int dk\ \sqrt{\frac{\pi}{a}}\exp\left(-\frac{k^2}{4a} + ikx - \frac{i\hbar k^2 t}{2m}\right), \\
&= \frac{1}{2\pi}\left(\frac{2a}{\pi}\right)^{\frac{1}{4}}\sqrt{\frac{1}{4a}}\frac{1}{\sqrt{1/4a + i\hbar t/2m}}\exp\left(\frac{-x^2}{4(1/4a + i\hbar t/2m)}\right), \\
&= \left(\frac{2a}{\pi}\right)^{\frac{1}{4}}\left(\frac{1}{\sqrt{1 + 2i\hbar a t/m}}\right)\exp\left(\frac{-ax^2}{1 + 2i\hbar a t/m}\right)_{\Box}
\end{aligned}$$

Using the definition of $\omega = \sqrt{a/[1+(2\hbar at/m)^2]}$, we can rewrite our answer as

$$|\Psi(x,t)|^2 = \sqrt{\frac{2}{\pi}}\omega e^{-2\omega^2x^2}.$$

For large t, we have that ω is inversely proportional to time, so the wave-function is of the same form but with its rms spread proportional to t. We can also notice that when written in this form, that $\langle x\rangle, \langle p\rangle = 0$. For $\langle x^2\rangle$, we integrate directly and find $1/4\omega^2$. $\langle p^2\rangle$ involves integration by parts as follows

$$\begin{aligned}
\langle p^2\rangle &= -\hbar^2 \int \Psi^* \nabla^2 \Psi, \\
&= +\hbar^2 \int \nabla\Psi^* \nabla\Psi, \\
&= \hbar^2 \sqrt{\frac{2}{\pi}}\omega \int \left(\frac{-2ax}{1-2i\hbar at/m}\right)\left(\frac{-2ax}{1+2i\hbar at/m}\right)\exp(-2\omega^2x^2), \\
&= \hbar^2\sqrt{\frac{1}{\pi}}4\ a\ \omega^3 \int x^2 \exp(-2\omega^2x^2), \\
&= \hbar^2 a. \qquad (6)
\end{aligned}$$

We see that $\sigma_x\sigma_p = \hbar\sqrt{a}/(2\omega)$, which when $t \to 0, \omega \to \sqrt{a}$, and $\sigma_x\sigma_p = \hbar/2$.

6. Current Vector

   Find the current density carried by a plane wave Ae^{ikx} in one dimension, showing that it is in fact what one would expect from the formula $\rho\mathbf{v}$, and verify that it satisfies the equation of continuity.

   [For $\psi = A\exp(ikx)$, We know that the current density is

$$\begin{aligned}
j &= \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right), \\
&= \frac{i\hbar}{2m}(-ikA^2 - ikA^2). \qquad (7)
\end{aligned}$$

   Therefore, $j = (\hbar k/m)(A^2) = (v)(\rho)$. The continuity equation $\partial_t\rho = -\nabla.\mathbf{j}$ is trivially satisfied.]

7. Commutators

   Prove the following:

$$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$$
$$[\hat{A} + \hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]$$
$$[a, \hat{A}] = 0$$
$$[a\hat{A}, \hat{B}] = a[\hat{A}, \hat{B}]$$
$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$$

   where a is a constant number.

   [Using the definition that $[A, B] = AB - BA$, it is straight forward to show these relations. We just do one of them here.

$$\begin{aligned} [AB, C] &= ABC - CAB = ABC + (-ACB + ACB) + (-ACB + ACB) - CAB \\ &= A[B, C] + (ACB - ACB) + [A, C]B \\ &= A[B, C] + [A, C]B \ \square \end{aligned} \tag{8}$$

8. Sum Rule

   (a) Consider the commutator $[H, x]$

$$\begin{aligned} [H, x] &= [\frac{p^2}{2m} + V(x), x] \\ &= \frac{1}{2m}[p^2, x] \\ &= \frac{1}{2m}\left(p[p, x] + [p, x]p\right) \quad \text{(where we have used the result of problem 7.)} \\ &= \frac{-i\hbar}{m}p \end{aligned}$$

   Then

$$\begin{aligned} \langle \psi_n \mid [H, x] \mid \psi_{n'} \rangle &= \langle \psi_n \mid [Hx - xH] \mid \psi_{n'} \rangle \\ -\frac{i\hbar}{m} \langle \psi_n \mid p \mid \psi_{n'} \rangle &= \langle \psi_n \mid (E_n x - xE_{n'}) \mid \psi_{n'} \rangle \\ &= (E_n - E_{n'}) \langle \psi_n \mid x \mid \psi_{n'} \rangle \\ \rightarrow \langle \psi_n \mid p \mid \psi_{n'} \rangle &= i\frac{m}{\hbar}(E_n - E_{n'}) \langle \psi_n \mid x \mid \psi_{n'} \rangle \end{aligned}$$

(b)

$$
\begin{aligned}
\left\langle \psi_n \mid p^2 \mid \psi_{n'} \right\rangle &= -\frac{m^2}{\hbar^2} \left\langle \psi_n \mid [H,x]^2 \mid \psi_{n'} \right\rangle \\
&= -\frac{m^2}{\hbar^2} \left\langle \psi_n \mid [H,x]^2 \mid \psi_{n'} \right\rangle && (9) \\
&= -\frac{m^2}{\hbar^2} \sum_{n'} \langle \psi_n \mid [H,x] \mid \psi_{n'} \rangle \langle \psi_{n'} \mid [H,x] \mid \psi_n \rangle && (10) \\
&= -\frac{m^2}{\hbar^2} \sum_{n'} (E_n - E_{n'}) \langle \psi_n \mid x \mid \psi_{n'} \rangle (E_{n'} - E_n) \langle \psi_{n'} \mid x \mid \psi_n \rangle \\
&= \frac{m^2}{\hbar^2} \sum_{n'} (E_n - E_{n'})^2 \langle \psi_n \mid x \mid \psi_{n'} \rangle \langle \psi_{n'} \mid x \mid \psi_n \rangle \\
&= \frac{m^2}{\hbar^2} \sum_{n'} (E_n - E_{n'})^2 | \langle \psi_n \mid x \mid \psi_{n'} \rangle |^2
\end{aligned}
$$

In going from Equation 9 to 10 we use the fact the $\mid \psi_{n'} \rangle$ is a complete set.

Integrals

$$\int x \sin(ax) \sin(bx) dx = \frac{1}{2} \left(\frac{\cos(a-b)x}{(a-b)^2} - \frac{\cos(a-b)x}{(a+b)^2} + \frac{x \sin(a-b)x}{a-b} - \frac{x \sin(a+b)x}{a+b} \right)$$