P443 HW #2 (problem 8 corrected 2/5)
Due February 6, 2008

1. Griffiths, 2.12. Find $\langle x\rangle, \langle p\rangle, \langle x^2\rangle, \langle p^2\rangle$, and $\langle T\rangle$, for the $n^{\text{th}}$ stationary state of the harmonics oscillator, using the method of Example 2.5, (namely in terms of $\langle a_-\rangle, \langle a_+\rangle, \langle a_-a_+\rangle$, etc.)

2. Griffiths 2.14. A particle is in the ground state of the harmonic oscillator with classical frequency $\omega$, when suddenly the spring constant quadrupoles, so $\omega' = 2\omega$, without initially changing the wave function (of course, $\Psi$ will now *evolve* differently, because the Hamiltonian has changed). What is the probability that a measurement of the energy would still return the value $\hbar\omega/2$? What is the probability of getting $\hbar\omega$? $[Answer : 0.943.]$

3. Griffiths 2.26. What is the Fourier transform of $\delta(x)$? Using Plancherel's theorem, show that
$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dk.$$

4. Griffiths 2.38. A particle of mass $m$ is in the ground state of the infinite square well
$$V(x) = \begin{cases} 0, & \text{if } 0 \le x \le a, \\ \infty, & \text{otherwise.} \end{cases} \tag{1}$$
Suddenly the well expands to twice its original size-the right wall moving from $a$ to $2a$-leaving the wave function (momentarily) undisturbed. The energy of the particle is now measured.

   (a) What is the most probable result? What is the probability of getting that result?

   (b) What is the *next* most probable result, and what is its probability?

   (c) What is the *expectation value* of the energy? $Hint$ : If you find yourself confronted with an infinite series, try another method.

5. Griffiths 3.23. The Hamiltonian for a certain two-level system is
$$\hat{H} = \epsilon(|\,1\rangle\langle 1\,| - |\,2\rangle\langle 2\,| + |\,1\rangle\langle 2\,| + |\,2\rangle\langle 1\,|),$$
where $|\,1\rangle, |\,2\rangle$ is an orthonormal basis and $\epsilon$ is a number with the dimensions of energy. Find its eigenvalues and eigenvectors (as linear combinations of $|\,1\rangle$ and $|\,2\rangle$). What is the matrix $\mathbf{H}$ representing $\hat{H}$ with respect to this basis?

6. Griffiths 3.24. Let $\hat{Q}$ be an operator with a complete set of orthonormal eigenvectors:

$$\hat{Q}|\, e_n\rangle = q_n|\, e_n\rangle \quad (n = 1, 2, 3, ...).$$

Show that $\hat{Q}$ can be written in terms of its **spectral decomposition:**

$$\hat{Q} = \sum_n q_n|\, e_n\rangle\langle e_n\, |.$$

*Hint* : An operator is characterized by its action on all possible vectors, what you must show is that

$$\hat{Q}|\, \alpha\rangle = \left\{ \sum_n q_n|\, e_n\rangle\langle e_n\, | \right\} |\, \alpha\rangle,$$

for any vector $|\, \alpha\rangle$.

7. Time dependence

Show that if $\hat{Q}$ is an operator that does not involve time explicitly, and if $\psi$ is any eigenfunction of $\hat{H}$, that the expectation value of $\hat{Q}$ in the state of $\psi$ is independent of time.

8. Collapse of the wave function

Consider a particle in the infinite square well potential from problem 4.

(a) Show that the stationary states are $\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$ and the energy spectrum is $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$ where the width of the box is $a$.

(b) Suppose we now make a measurement that locates a particle initially in state $\psi_n(x)$ so that it is now in the position $a/2 - \epsilon/2 \leq x \leq a/2 + \epsilon/2$ and described by the state $\alpha$. In the limit where $\epsilon \ll a$, the result of the measurement projects the system onto a superposition of eigenstates of energy. The probability of finding the particle in any eigenstate is $P(E_n) = |\langle\psi_n \mid \alpha\rangle|^2$. A reasonable estimate of the state $|\, \alpha\rangle$ is $\psi_\alpha(x) = \sqrt{\epsilon}\delta^\epsilon(x - a/2)$ where $\delta^{(\epsilon)}(x - a/2) = 1/\epsilon$ for $a/2 - \epsilon/2 \leq x \leq a/2 + \epsilon/2$ and $\delta^\epsilon(x - \frac{a}{2}) = 0$ everywhere else. Calculate the probability $P(E_n)$.