Physics 443, Solutions to PS 3[1]

1. Griffiths 3.23. It is easiest to first write the hamiltonian matrix. By inspection

$$H = \epsilon \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

We find the eigenvalues λ by setting

$$\det(H - \lambda I) = 0$$

Then $\lambda_\pm = \pm\epsilon\sqrt{2}$. Let

$$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$$

be an eigenvector. Then

$$\begin{aligned} H\vec{v}_\pm &= \lambda_\pm \vec{v}_\pm \\ &= \epsilon \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \lambda_\pm \begin{pmatrix} a \\ b \end{pmatrix} \\ \Rightarrow \quad \vec{v}_\pm &= \begin{pmatrix} 1 \\ -1 \pm \sqrt{2} \end{pmatrix} \end{aligned}$$

Or in the $|\,\rangle$ representation

$$|\, v_\pm\rangle = |\, 1\rangle + (-1 \pm \sqrt{2})|\, 2\rangle$$

2. Griffiths 3.24.

Since the set of orthonormal vectors $|\, e_n\rangle$ is complete, any state can be written as a linear combination of those vectors. In paricular, the state $|\, \alpha\rangle$ can be written as

$$|\, \alpha\rangle = \sum_m a_m |\, e_m\rangle \tag{1}$$

Then

$$\langle e_n \,|\, \alpha\rangle = \sum_m a_m \langle e_n \,|\, e_m\rangle = \sum_m a_m \delta_{mn} = a_n \tag{2}$$

where we have used the orhthonormality of the eigenvectors. Finally substitute $\langle e_n \,|\, \alpha\rangle = a_n$ from Equation 3 into Equation 2.

$$|\, \alpha\rangle = \sum_n |\, e_n\rangle\langle e_n \,|\, \alpha\rangle \tag{3}$$

[1] Courtesy Shaffique Adam

So

$$\begin{aligned}\hat{Q}|\,\alpha\rangle &= \sum_n \hat{Q}|\,e_n\rangle\langle e_n \mid \alpha\rangle \\ &= \sum_n q_n|\,e_n\rangle\langle e_n \mid \alpha\rangle \\ &= \left(\sum_n q_n|\,e_n\rangle\langle e_n\,|\right)|\,\alpha\rangle\end{aligned}$$

3. Griffiths 3.3. We are given that $\left\langle h \mid \hat{Q} \mid h\right\rangle = \left\langle \hat{Q}h \mid h\right\rangle$ for all states $|\,h\rangle$. If we define $|\,h\rangle = |\,f\rangle + |\,g\rangle$ then

$$\begin{aligned}&\left\langle f \mid \hat{Q} \mid f\right\rangle + \left\langle f \mid \hat{Q} \mid g\right\rangle + \left\langle g \mid \hat{Q} \mid f\right\rangle + \left\langle g \mid \hat{Q} \mid g\right\rangle \\ &\quad = \left\langle \hat{Q}f \mid f\right\rangle + \left\langle \hat{Q}f \mid g\right\rangle + \left\langle \hat{Q}g \mid f\right\rangle + \left\langle \hat{Q}g \mid g\right\rangle\end{aligned} \tag{4}$$

By hypothesis $\left\langle f \mid \hat{Q} \mid f\right\rangle = \left\langle \hat{Q}f \mid f\right\rangle$ and similarly for $|\,g\rangle$ so Equation 3 reduces to

$$\left\langle f \mid \hat{Q} \mid g\right\rangle + \left\langle g \mid \hat{Q} \mid f\right\rangle = \left\langle \hat{Q}f \mid g\right\rangle + \left\langle \hat{Q}g \mid f\right\rangle \tag{5}$$

Alternatively, if we let $|\,h\rangle = |\,f\rangle + i|\,g\rangle$ we find that

$$i\left\langle f \mid \hat{Q} \mid g\right\rangle - i\left\langle g \mid \hat{Q} \mid f\right\rangle = i\left\langle \hat{Q}f \mid g\right\rangle - i\left\langle \hat{Q}g \mid f\right\rangle \tag{6}$$

or

$$\left\langle f \mid \hat{Q} \mid g\right\rangle - \left\langle g \mid \hat{Q} \mid f\right\rangle = \left\langle \hat{Q}f \mid g\right\rangle - \left\langle \hat{Q}g \mid f\right\rangle \tag{7}$$

The sum of equations (4) and (6) gives $\left\langle f \mid \hat{Q} \mid g\right\rangle = \left\langle \hat{Q}g \mid f\right\rangle$ and the difference gives $\left\langle g \mid \hat{Q} \mid f\right\rangle = \left\langle \hat{Q}g \mid f\right\rangle$

4. Griffiths 3.31. We have:

$$\begin{aligned}\frac{d}{dt}\langle xp\rangle &= \frac{i}{\hbar}\langle[\frac{p^2}{2m} + V(x), xp]\rangle, \\ &= \frac{i}{\hbar}\langle[\frac{p^2}{2m}, xp] + [V(x), xp] + \rangle, \\ &= \frac{i}{\hbar}\langle[\frac{pp}{2m}, x]p + x[V(x), p] + \rangle, \\ &= \frac{i}{\hbar}\langle -2i\hbar\frac{p^2}{2m} + x(\hbar i\frac{\partial V}{\partial x})\rangle, \\ &= 2\langle T\rangle - \langle x\frac{\partial V}{\partial x}\rangle.\end{aligned} \tag{8}$$

For a stationary state, we see that $2\langle T\rangle = \langle x\partial_x V\rangle$, which is the **Virial Theorem**. For the Harmonic Oscillator $V(x) = m\omega^2x^2/2$, using the Virial Theorem, we see that $\langle T\rangle = \langle m\omega^2 x^2/2\rangle = \langle V(x)\rangle$.

5. Griffiths 3.33. We begin by using that

$$
\begin{aligned}
a_{\pm} &= \frac{1}{\sqrt{2m}}(P \pm im\omega x), \\
P &= \sqrt{\frac{m}{2}}(a_+ + a_-), \\
x &= \frac{-i}{\sqrt{2m\omega}}(a_+ - a_-), \\
a_+|n-1\rangle &= i\sqrt{n\hbar\omega}|n\rangle, \\
a_-|n\rangle &= -i\sqrt{n\hbar\omega}|n-1\rangle, \\
\langle n|x|n'\rangle &= \frac{-i}{\sqrt{2m\omega}}(\langle n|a_+ - a_-|n'\rangle), \\
&= \frac{-i}{\sqrt{2m\omega}}(\delta_{n',n-1}i\sqrt{n\hbar\omega} - \delta_{n,n'-1}(-i\sqrt{n'\hbar\omega})), \\
&= \sqrt{\frac{\hbar}{2m\omega}}(\delta_{n',n-1}\sqrt{n} + \delta_{n,n'-1}\sqrt{n'}), \\
\langle n|p|n'\rangle &= \sqrt{\frac{m}{2}}(\langle n|a_+ + a_-|n'\rangle), \\
&= i\sqrt{\frac{\hbar m\omega}{2}}(\delta_{n',n-1}\sqrt{n} - \delta_{n,n'-1}\sqrt{n'}).
\end{aligned}
$$

We can then write these out in matrix notation as

$$
\begin{aligned}
x &= \sqrt{\frac{\hbar}{2m\omega}}\begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & \ldots \\ \sqrt{2} & 0 & \sqrt{3} & 0 & \ldots \\ 0 & \sqrt{3} & 0 & \sqrt{4} & \ldots \\ 0 & 0 & \sqrt{4} & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \\
p &= i\sqrt{\frac{\hbar m\omega}{2}}\begin{pmatrix} 0 & -\sqrt{2} & 0 & 0 & \ldots \\ \sqrt{2} & 0 & -\sqrt{3} & 0 & \ldots \\ 0 & \sqrt{3} & 0 & -\sqrt{4} & \ldots \\ 0 & 0 & \sqrt{4} & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.
\end{aligned} \tag{9}
$$

And you can verify that $p^2/2m+(m\omega^2/2)x^2$ is diagonal with the matrix element given by $\hbar\omega(n+1/2)$.

6. Griffiths 3.38.

   (a) By inspection the eignenvalues of $\mathbf{H}$ are $E_1 = \hbar\omega, E_2 = E_3 = 2\hbar\omega$ and the eigenvectors are

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

   To find the eigenvalues,γ, we set

$$\begin{aligned} \det(\mathbf{A} - \gamma\mathbf{I}) &= \begin{vmatrix} \gamma & \lambda & 0 \\ \lambda & \gamma & 0 \\ 0 & 0 & 2\lambda - \gamma \end{vmatrix} = 0 \\ &\Rightarrow \gamma^2(2\lambda-\gamma) - \lambda^2(2\lambda-\gamma) = 0 \end{aligned}$$

   Then the normalized eigenvalues of $\mathbf{A}$ are $A_1 = \lambda,\ A_2 = -\lambda,\ A_3 = 2\lambda$ and the eigenvectors are

$$a_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad a_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

   The same strategy gives the eigenvalues $\mathbf{B}$ are $B_1 = 2\mu,\ B_2 = \mu,\ B_3 = -\mu$ and the eigenvectors are

$$b_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad b_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad b_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

   (b)

$$\begin{aligned} \langle H \rangle &= \langle S \mid \mathbf{H} \mid S \rangle \\ &= \hbar\omega \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\ &= \hbar\omega(|c_1|^2 + 2(|c_2|^2 + |c_3|^2)) \\ &= \hbar\omega(2 - |c_1|^2) \end{aligned}$$

$$
\begin{aligned}
\langle A \rangle &= \lambda \,(c_1 \quad c_2 \quad c_3) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\
&= \lambda(c_1^* c_2 + c_2^* c_1 + 2|c_3|^2)
\end{aligned}
$$

$$
\begin{aligned}
\langle B \rangle &= \mu \,(c_1 \quad c_2 \quad c_3) \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\
&= \mu(2|c_1|^2 + c_2^* c_3 + c_3^* c_2)
\end{aligned}
$$

(c)

$$
\mid S(t)\rangle = c_1 e^{-i\omega t} + c_2 e^{-2i\omega t} + c_3 e^{-2i\omega t}
$$

The probability of measuring energies E_1, E_2 and E_3 is $|c_1|^2$, $|c_2|^2$, and $|c_3|^2$ respectively independent of time. The probability of measuring A_i is $|\langle a_i \mid S(t)\rangle|^2$

$$
\begin{aligned}
|\langle a_1 \mid S(t)\rangle|^2 &= |\frac{1}{\sqrt{2}}\,(1 \quad 1 \quad 0) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} |^2 = \frac{1}{2}|c_1 e^{-i\omega t} + c_2 e^{-2i\omega t}|^2 \\
|\langle a_2 \mid S(t)\rangle|^2 &= |\frac{1}{\sqrt{2}}\,(1 \quad -1 \quad 0) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} |^2 = \frac{1}{2}|c_1 e^{-i\omega t} - c_2 e^{-2i\omega t}|^2 \\
|\langle a_3 \mid S(t)\rangle|^2 &= |\,(0 \quad 0 \quad 1) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} |^2 = |c_3|^2
\end{aligned}
$$

The probability of measuring B_i is $|\langle b_i \mid S(t)\rangle|^2$

$$
\begin{aligned}
|\langle b_1 \mid S(t)\rangle|^2 &= |\,(1 \quad 0 \quad 0) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} |^2 = \frac{1}{2}|c_1|^2 \\
|\langle b_2 \mid S(t)\rangle|^2 &= |\frac{1}{\sqrt{2}}\,(0 \quad 1 \quad 1) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} |^2 = \frac{1}{2}|c_2 e^{-2i\omega t} + c_3 e^{-2i\omega t}|^2 = \frac{1}{2}|c_2 + c_3|^2 \\
|\langle b_3 \mid S(t)\rangle|^2 &= |\frac{1}{\sqrt{2}}\,(0 \quad 1 \quad -1) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} |^2 = \frac{1}{2}|c_2 e^{-2i\omega t} - c_3 e^{-2i\omega t}|^2 = \frac{1}{2}|c_2 - c_3|^2
\end{aligned}
$$

7. Charmonium. The Schrodinger equation for charmonium is

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \alpha r\psi = E\psi$$

Define $u(r) = r\psi$ and for spherically syymetric wave functions, the Schrodinger equation reduces to

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \alpha ru = Eu$$

Let $r = l_0 z$ and $E = \epsilon E_0$ where z and ϵ are dimensionless and the Schrodinger equation becomes

$$-\frac{\hbar^2}{2ml_0^2}\frac{d^2u}{dz^2} + \alpha l_0 zu = \epsilon E_0 u$$

or

$$-\frac{d^2u}{dz^2} + \alpha\frac{2ml_0^2}{\hbar^2}l_0 zu = \epsilon\frac{2ml_0^2E_0}{\hbar^2}u$$

Set $l_0 = \left(\frac{\hbar^2}{2m\alpha}\right)^{1/3}$ and $E_0 = \frac{\hbar^2}{2ml_0^2} = \left(\frac{\hbar^2\alpha^2}{2m}\right)^{1/3}$ and our differential equation looks like

$$-\frac{d^2u}{dz^2} + zu = \epsilon u$$

Now let $y = z - \epsilon$ and we have Airy's equation

$$-\frac{d^2u}{dy^2} + yu = 0$$

Since $u(r) = \psi(r)/r$,then it must be that $u(0) = 0$ so that $\psi(0)$ is finite. Therefore $u(z - \epsilon) = u(-\epsilon) = 0$. The energy eigenvalues, ϵ are the zeros of the Airy function a_i. The first two zeros are 2.3 and 4.1 so $E_1 = 2.3E_0$ and $E_2 = 4.1E_0$.

We have that

$$\begin{aligned} m_{1s}c^2 &= 2m_cc^2 + E_1 \\ m_{2s}c^2 &= 2m_cc^2 + E_2 \end{aligned} \tag{10}$$

where m_c is the charmed quark mass and E_1 and E_2 are the binding energies. The difference of the two equations yields $E_0 = (m_{2s} -$

$m_{1s})c^2/1.8 = 0.325GeV$. We find $m_c c^2 = 1.176GeV$. Finally the reduced mass $m = m_c/2$. Meanwhile,

$$E_0 = \left(\frac{\hbar^2\alpha^2}{2m}\right)^{1/3}$$

$$\Rightarrow \alpha = \frac{2mE_0^3}{\hbar^2} = \frac{m_c c^2 E_0^3}{\hbar^2 c^2} = \frac{(1.176GeV)(0.325GeV)^3}{(.197GeV - fm)^2} = 1.02GeV/fm$$

And

$$l_0 = \left(\frac{\hbar^2 c^2}{m_c c^2 \alpha}\right)^{1/3} = \left(\frac{(0.197GeV - fm)^2}{(1.176GeV)(1.02GeV/fm)}\right)^{1/3} = 0.318fm.$$

8. Rotations. We define $x = iL_y\theta/\hbar$. Notice that

$$\begin{aligned} x &= \frac{\theta}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ x^2 &= -\left(\frac{\theta}{2}\right)^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ x^3 &= -\left(\frac{\theta}{2}\right)^3\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ x^2 &= \left(\frac{\theta}{2}\right)^4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

In particular

$$\begin{aligned} R(\theta) = e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ..., \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\left(1 - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + ...\right) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left(\left(\frac{\theta}{2}\right) - \frac{1}{3!}\left(\frac{\theta}{2}\right)^3 + ...\right), \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(\frac{\theta}{2}) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\sin(\frac{\theta}{2}), \\ &= \begin{pmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ -\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}. \end{aligned}$$

You can see that $L_y^\dagger = L_y$ and $R(\theta)^T R(\theta) = 1$, making L_y Hermitian and $R(\theta)$ unitary.